ECS 315: Probability and Random Processes **2016/1**

HW 14 — Due: Not Due

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Problem 1. Let a continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the probability density function of X is

$$f_X(x) = \begin{cases} 5, & 4.9 \le x \le 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability that a current measurement is less than 5 milliamperes.

(b) Find and plot the cumulative distribution function of the random variable X.

(c) Find the expected value of X.

(d) Find the variance and the standard deviation of X.

(e) Find the expected value of power when the resistance is 100 ohms?

Problem 2. The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the probability density function of X.

(b) What proportion of reactions is complete within 200 milliseconds?

Problem 3. Let $X \sim \mathcal{E}(5)$ and $Y = 2/X$.

(a) Check that Y is still a continuous random variable.

(b) Find $F_Y(y)$.

(c) Find $f_Y(y)$.

(d) (optional) Find $\mathbb{E}Y$. Hint: Because $\frac{d}{dy}e^{-\frac{10}{y}} = \frac{10}{y^2}e^{-\frac{10}{y}} > 0$ for $y \neq 0$. We know that

$e^{-\frac{10}{y}}$ is an increasing function on our range of integration. In particular, consider $y > 10/\ln(2)$. Then, $e^{-\frac{10}{y}} > \frac{1}{2}$. Hence,

$$\int_0^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}\frac{1}{2}dy = \int_{10/\ln 2}^\infty \frac{5}{y}dy.$$

Remark: To be technically correct, we should be a little more careful when writing $Y = \frac{2}{X}$ because it is undefined when $X = 0$. Of course, this happens with 0 probability; so it won't create any serious problem. However, to avoid the problem, we may define Y by

$$Y = \begin{cases} 2/X, & X \neq 0, \\ 0, & X = 0. \end{cases} \tag{14.1}$$

Problem 4. In wireless communications systems, fading is sometimes modeled by ***lognormal*** random variables. We say that a positive random variable Y is lognormal if $\ln Y$ is a normal random variable (say, with expected value m and variance σ^2).

Hint: First, recall that the ln is the natural log function (log base e). Let $X = \ln Y$. Then, because Y is lognormal, we know that $X \sim \mathcal{N}(m, \sigma^2)$. Next, write Y as a function of X.

(a) Check that Y is still a continuous random variable.

(b) Find the pdf of Y.

Problem 5. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

x \ y	2	4	5
1	0.02	0.10	0.08
3	0.08	0.32	0.40

(a) Evaluate the following quantities:

(i) The marginal pmf $p_X(x)$

(ii) The marginal pmf $p_Y(y)$

(iii) $\mathbb{E}X$

(iv) $\operatorname{Var} X$

(v) $\mathbb{E}Y$

(vi) $\operatorname{Var} Y$

(vii) $P[XY < 6]$

(viii) $P[X = Y]$

(ix) $\mathbb{E}[XY]$

(x) $\mathbb{E}[(X-3)(Y-2)]$

(xi) $\mathbb{E}[X(Y^3-11Y^2+38Y)]$

(xii) $\text{Cov}[X,Y]$

(xiii) $\rho_{X,Y}$

(b) Find $\rho_{X,X}$

(c) Calculate the following quantities using the values of $\text{Var}\, X$, $\text{Cov}[X,Y]$, and $\rho_{X,Y}$ that you got earlier.

(i) $\text{Cov}[3X+4, 6Y-7]$

(ii) $\rho_{3X+4,6Y-7}$

(iii) $\text{Cov}[X, 6X-7]$

(iv) $\rho_{X,6X-7}$

Problem 6. Suppose $X \sim \text{binomial}(5, 1/3)$, $Y \sim \text{binomial}(7, 4/5)$, and $X \perp\!\!\!\perp Y$. Evaluate the following quantities.

(a) $\mathbb{E}\left[(X-3)(Y-2)\right]$

(b) $\text{Cov}\left[X, Y\right]$

(c) $\rho_{X,Y}$

Extra Questions

Here are some extra questions for those who want more practice.

Problem 7. Consider a random variable X whose pdf is given by

$$f_X(x) = \begin{cases} cx^2, & x \in (1,2), \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = 4\left|X - 1.5\right|$.

(a) Find $\mathbb{E}Y$.

(b) Find $f_Y(y)$.

Problem 8. A webpage server can handle r requests per day. Find the probability that the server gets more than r requests at least once in n days. Assume that the number of requests on day i is $X_i \sim \mathcal{P}(\alpha)$ and that $X_1, \ldots, X_n$ are independent.

Problem 9. Suppose $X \sim$ binomial$(5, 1/3)$, $Y \sim$ binomial$(7, 4/5)$, and $X \perp\!\!\!\perp Y$.

(a) A vector describing the pmf of X can be created by the MATLAB expression:

```
x = 0:5; pX = binopdf(x,5,1/3).
```

What is the expression that would give `pY`, a corresponding vector describing the pmf of Y?

(b) Use `pX` and `pY` from part (a), how can you create the joint pmf matrix in MATLAB? Do not use "for-loop", "while-loop", "if statement". Hint: Multiply them in an appropriate orientation.

(c) Use `MATLAB` to evaluate the following quantities. Again, do not use "for-loop", "while-loop", "if statement".

(i) $\mathbb{E}X$

(ii) $P[X = Y]$

(iii) $P[XY < 6]$

Problem 10. Suppose $\operatorname{Var} X = 5$. Find $\operatorname{Cov}[X, X]$ and $\rho_{X,X}$.

Problem 11. Suppose we know that $\sigma_X = \frac{\sqrt{21}}{10}$, $\sigma_Y = \frac{4\sqrt{6}}{5}$, $\rho_{X,Y} = -\frac{1}{\sqrt{126}}$.

(a) Find $\operatorname{Var}[X + Y]$.

(b) Find $\mathbb{E}\left[(Y - 3X + 5)^2\right]$. Assume $\mathbb{E}\left[Y - 3X + 5\right] = 1$.

Problem 12. The input X and output Y of a system subject to random perturbations are described probabilistically by the joint pmf $p_{X,Y}(x, y)$, where $x = 1, 2, 3$ and $y = 1, 2, 3, 4, 5$. Let P denote the joint pmf matrix whose i,j entry is $p_{X,Y}(i, j)$, and suppose that

$$P = \frac{1}{71}\begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

(a) Find the marginal pmfs $p_X(x)$ and $p_Y(y)$.

(b) Find $\mathbb{E}X$

(c) Find $\mathbb{E}Y$

(d) Find $\operatorname{Var} X$

(e) Find $\operatorname{Var} Y$