ECS 315: Probability and Random Processes **2016/1**

HW 13 — Due: Dec 6, 5 PM

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Instructions

(a) This assignment has 8 pages.

(b) (1 pt) Write your first name and the last three digit of your student ID on the upper-right corner of ***every*** submitted sheet.

(c) (1 pt) For each part, write your explanation/derivation and answer in the space provided.

(d) (8 pt) It is important that you try to solve all non-optional problems.

(e) Late submission will be heavily penalized.

Problem 1. A random variable X is a Gaussian random variable if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2},$$

for some constant m and positive number σ. Furthermore, when a Gaussian random variable has $m = 0$ and $\sigma = 1$, we say that it is a standard Gaussian random variable. There is no closed-form expression for the cdf of the standard Gaussian random variable. The cdf itself is denoted by Φ and its values (or its complementary values $Q(\cdot) = 1 - \Phi(\cdot)$) are traditionally provided by a table.

Suppose Z is a standard Gaussian random variable.

(a) Use the Φ table to find the following probabilities:

(i) $P[Z < 1.52]$

(ii) $P[Z < -1.52]$

(iii) $P[Z > 1.52]$

(iv) $P[Z > -1.52]$

(v) $P[-1.36 < Z < 1.52]$

(b) Use the Φ table to find the value of c that satisfies each of the following relation.

(i) $P[Z > c] = 0.14$

(ii) $P[-c < Z < c] = 0.95$

Problem 2. The peak temperature T, as measured in degrees Fahrenheit, on a July day in New Jersey is a $\mathcal{N}(85, 100)$ random variable.

Remark: Do not forget that, for our class, the second parameter in $\mathcal{N}(\cdot, \cdot)$ is the variance (not the standard deviation).

(a) Express the cdf of T in terms of the Φ function.

(b) Express each of the following probabilities in terms of the Φ function(s). Make sure that the arguments of the Φ functions are positive. (Positivity is required so that we can directly use the Φ/Q tables to evaluate the probabilities.)

(i) $P[T > 100]$

(ii) $P[T < 60]$

(iii) $P[70 \leq T \leq 100]$

(c) Express each of the probabilities in part (b) in terms of the Q function(s). Again, make sure that the arguments of the Q functions are positive.

(i) $P[T > 100]$

(ii) $P[T < 60]$

(iii) $P[70 \leq T \leq 100]$

(d) Evaluate each of the probabilities in part (b) using the Φ/Q tables.

(i) $P[T > 100]$

(ii) $P[T < 60]$

(iii) $P[70 \leq T \leq 100]$

(e) Observe that the Φ table ("Table 4" from the lecture) stops at $z = 2.99$ and the Q table ("Table 5" from the lecture) starts at $z = 3.00$. Why is it better to give a table for $Q(z)$ instead of $\Phi(z)$ when z is large?

Problem 3. Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

[Montgomery and Runger, 2010, Q4-97]

Problem 4. Consider each random variable X defined below. Let $Y = 1 + 2X$. (i) Find and sketch the pdf of Y and (ii) Does Y belong to any of the (popular) families discussed in class? If so, state the name of the family and find the corresponding parameters.

(a) $X \sim \mathcal{U}(0, 1)$

(b) $X \sim \mathcal{E}(1)$

(c) $X \sim \mathcal{N}(0,1)$

Problem 5. Consider each random variable X defined below. Let $Y = 1 - 2X$. (i) Find and sketch the pdf of Y and (ii) Does Y belong to any of the (popular) families discussed in class? If so, state the name of the family and find the corresponding parameters.

(a) $X \sim \mathcal{U}(0,1)$

(b) $X \sim \mathcal{E}(1)$

(c) $X \sim \mathcal{N}(0,1)$

Problem 6. Let $X \sim \mathcal{E}(3)$.

(a) For each of the following function $g(x)$. Indicate whether the random variable $Y = g(X)$ is a continuous random variable.

(i) $g(x) = x^2$.

(ii) $g(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$.

(iii) $g(x) = \begin{cases} 4e^{-4x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

(iv) $g(x) = \begin{cases} x, & x \leq 5, \\ 5, & x > 5. \end{cases}$

(b) Repeat part (a), but now check whether the random variable $Y = g(X)$ is a discrete random variable.

(i) $g(x) = x^2$.

(ii) $g(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$.

(iii) $g(x) = \begin{cases} 4e^{-4x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

(iv) $g(x) = \begin{cases} x, & x \leq 5, \\ 5, & x > 5. \end{cases}$

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Extra Questions

Here are some optional questions for those who want more practice.

Problem 7. Cholesterol is a fatty substance that is an important part of the outer lining (membrane) of cells in the body of animals. Its normal range for an adult is 120–240 mg/dl. The Food and Nutrition Institute of the Philippines found that the total cholesterol level for Filipino adults has a mean of 159.2 mg/dl and 84.1% of adults have a cholesterol level below 200 mg/dl. Suppose that the cholesterol level in the population is normally distributed.

(a) Determine the standard deviation of this distribution.

(b) What is the value of the cholesterol level that exceeds 90% of the population?

(c) An adult is at moderate risk if cholesterol level is more than one but less than two standard deviations above the mean. What percentage of the population is at moderate risk according to this criterion?

(d) An adult is thought to be at high risk if his cholesterol level is more than two standard deviations above the mean. What percentage of the population is at high risk?

Problem 8 (Q3.5.6)**.** Solve this question using the Φ/Q table.

A professor pays 25 cents for each blackboard error made in lecture to the student who points out the error. In a career of n years filled with blackboard errors, the total amount in dollars paid can be approximated by a Gaussian random variable Y_n with expected value $40n$ and variance $100n$.

(a) What is the probability that Y_{20} exceeds 1000?

(b) How many years n must the professor teach in order that $P[Y_n > 1000] > 0.99$?