ECS 315: Probability and Random Processes **2016/1**

HW Solution 12 — Due: November 29, 5 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1 (Yates and Goodman, 2005, Q3.3.4)**.** The pdf of random variable Y is

$$f_Y(y) = \begin{cases} y/2 & 0 \le y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{E}[Y]$.

(b) Find $\operatorname{Var} Y$.

Solution:

(a) Recall that, for continuous random variable Y,

$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y) dy.$$

Note that when y is outside of the interval $[0, 2)$, $f_Y(y) = 0$ and hence does not affect the integration. We only need to integrate over $[0, 2)$ in which $f_Y(y) = \frac{y}{2}$. Therefore,

$$\mathbb{E}Y = \int_0^2 y\left(\frac{y}{2}\right)dy = \int_0^2 \frac{y^2}{2}dy = \left.\frac{y^3}{2 \times 3}\right|_0^2 = \boxed{\frac{4}{3}}.$$

(b) The variance of any random variable Y (discrete or continuous) can be found from

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2$$

We have already calculate $\mathbb{E}Y$ in the previous part. So, now we need to calculate $\mathbb{E}[Y^2]$. Recall that, for continuous random variable,

$$\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y) dy.$$

Here, $g(y) = y^2$. Therefore,

$$\mathbb{E}\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y) dy.$$

Again, in the integration, we can ignore the y whose $f_Y(y) = 0$:

$$\mathbb{E}\left[Y^2\right] = \int_0^2 y^2 \left(\frac{y}{2}\right) dy = \int_0^2 \frac{y^3}{2} dy = \left.\frac{y^4}{2 \times 4}\right|_0^2 = \boxed{2}.$$

Plugging this into the variance formula gives

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \boxed{\frac{2}{9}}.$$

Problem 2 (Yates and Goodman, 2005, Q3.3.6)**.** The cdf of random variable V is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144, & -5 \le v < 7, \\ 1 & v \ge 7. \end{cases}$$

(a) What is $f_V(v)$?

(b) What is $\mathbb{E}[V]$?

(c) What is $\operatorname{Var}[V]$?

(d) What is $\mathbb{E}[V^3]$?

Solution: First, let's check whether V is a continuous random variable. This can be done easily by checking whether its cdf $F_V(v)$ is a continuous function. The cdf of V is defined using three expressions. Note that each expression is a continuous function. So, we only need to check whether there is/are any jump(s) at the boundaries: $v = 5$ and $v = 7$. Plugging $v = 5$ into $(v+5)^2/144$ gives 0 which matches the value of the expression for $v < -5$. Plugging $v = 7$ into $(v+5)^2/144$ gives 1 which matches the value of the expression for $v \ge 7$. SO, there is no discontinuity in $F_V(v)$. It is a continuous function and hence V itself is a continuous random variable.

(a) We can find the pdf $f_V(v)$ at almost all of the v by finding the derivative of the cdf $F_V(v)$:

$$f_V(v) = \frac{d}{dv} F_V(v) = \begin{cases} 0, & v < -5, \\ \frac{v+5}{72}, & -5 < v < 7, \\ 0, & v > 7. \end{cases}$$

Note that we still haven't specified $f_V(v)$ at $v = 5$ and $v = 7$. This is because the formula for $F_V(v)$ changes at those points and hence to actually find the derivatives, we would need to look at both the left and right derivatives at these points. The derivative may not even exist there. The good news is that we don't have to actually find them because $v = 5$ and $v = 7$ correspond to just two points on the pdf. Because V is a continuous random variable, we can "define" or "set " $f_V(v)$ to be any values there. In this case, for brevity of the expression, let's set the pdf to be 0 there. This gives

$$f_V(v) = \frac{d}{dv}F_V(v) = \boxed{\begin{cases} \frac{v+5}{72}, & -5 < v < 7, \\ 0, & \text{otherwise.} \end{cases}}$$

(b) $\mathbb{E}[V] = \int_{-\infty}^{\infty} v f_V(v)\, dv = \int_{-5}^{7} v\left(\frac{v+5}{72}\right) dv = \frac{1}{72}\int_{-5}^{7} v^2 + 5v dv = \boxed{3}$.

(c) $\mathbb{E}[V^2] = \int_{-\infty}^{\infty} v^2 f_V(v)\, dv = \int_{-5}^{7} v^2\left(\frac{v+5}{72}\right) dv = 17.$
Therefore, $\operatorname{Var} V = \mathbb{E}[V^2] - (\mathbb{E}[V])^2 = 17 - 9 = \boxed{8}$.

(d) $\mathbb{E}[V^3] = \int_{-\infty}^{\infty} v^3 f_V(v)\, dv = \int_{-5}^{7} v^3\left(\frac{v+5}{72}\right) dv = \boxed{\frac{431}{5} = 86.2}$.

Problem 3 (Yates and Goodman, 2005, Q3.4.5)**.** X is a continuous uniform RV on the interval $(-5, 5)$.

(a) What is its pdf $f_X(x)$?

(b) What is its cdf $F_X(x)$?

(c) What is $\mathbb{E}[X]$?

(d) What is $\mathbb{E}[X^5]$?

(e) What is $\mathbb{E}\left[e^X\right]$?

Solution: For a uniform random variable X on the interval (a, b), we know that

$$f_X(x) = \begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{1}{b-a}, & a \le x \le b \end{cases}$$

and

$$F_X(x) = \begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{x-a}{b-a}, & a \le x \le b. \end{cases}$$

In this problem, we have $a = -5$ and $b = 5$.

(a) $f_X(x) = \boxed{\begin{cases} 0, & x < -5 \text{ or } x > 5, \\ \frac{1}{10}, & -5 \le x \le 5 \end{cases}}$

(b) $F_X(x) = \boxed{\begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{x+5}{10}, & a \le x \le b. \end{cases}}$

(c) $\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-5}^{5} x \times \frac{1}{10} dx = \frac{1}{10} \left. \frac{x^2}{2} \right|_{-5}^{5} = \frac{1}{20}\left(5^2 - (-5)^2\right) = \boxed{0}$.

In general,

$$\mathbb{E}X = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{a+b}{2}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}X = \boxed{0}$.

(d) $\mathbb{E}[X^5] = \int_{-\infty}^{\infty} x^5 f_X(x)\,dx = \int_{-5}^{5} x^5 \times \frac{1}{10} dx = \frac{1}{10} \left. \frac{x^6}{6} \right|_{-5}^{5} = \frac{1}{60}\left(5^6 - (-5)^6\right) = \boxed{0}$.

In general,

$$\mathbb{E}\left[X^5\right] = \int_a^b x^5 \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x^5 dx = \frac{1}{b-a} \left. \frac{x^6}{6} \right|_a^b = \frac{1}{b-a} \frac{b^6 - a^6}{2}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}[X^5] = \boxed{0}$.

(e) In general,

$$\mathbb{E}\left[e^X\right] = \int_a^b e^x \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b e^x dx = \frac{1}{b-a} \left. e^x \right|_a^b = \frac{e^b - e^a}{b-a}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}\left[e^X\right] = \boxed{\frac{e^5 - e^{-5}}{10}} \approx 14.84$.

Problem 4 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

Solution: First, because Θ is a uniform random variable on the interval $(0, 2\pi)$, we know that $f_\Theta(\theta) = \frac{1}{2\pi}1_{(0,2\pi)}(t)$. Therefore, for "any" function g, we have

$$\mathbb{E}[g(\Theta)] = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta) d\theta.$$

(a) X is a function of Θ. $\mathbb{E}[X] = 5\mathbb{E}[\cos(7t + \Theta)] = 5\int_0^{2\pi} \frac{1}{2\pi}\cos(7t + \theta)d\theta$. Now, we know that integration over a cycle of a sinusoid gives 0. So, $\mathbb{E}[X] = \boxed{0}$.

(b) Y is another function of Θ.

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)] = \int_0^{2\pi} \frac{1}{2\pi} 5\cos(7t_1 + \theta) \times 5\cos(7t_2 + \theta)d\theta \\ &= \frac{25}{2\pi}\int_0^{2\pi} \cos(7t_1 + \theta) \times \cos(7t_2 + \theta)d\theta.\end{aligned}$$

Recall[1] the cosine identity

$$\cos(a) \times \cos(b) = \frac{1}{2}\left(\cos(a + b) + \cos(a - b)\right).$$

Therefore,

$$\begin{aligned}\mathbb{E}Y &= \frac{25}{4\pi}\int_0^{2\pi} \cos(14t + 2\theta) + \cos(7(t_1 - t_2))\, d\theta \\ &= \frac{25}{4\pi}\left(\int_0^{2\pi} \cos(14t + 2\theta)\, d\theta + \int_0^{2\pi} \cos(7(t_1 - t_2))\, d\theta\right).\end{aligned}$$

The first integral gives 0 because it is an integration over two period of a sinusoid. The integrand in the second integral is a constant. So,

$$\mathbb{E}Y = \frac{25}{4\pi}\cos(7(t_1 - t_2))\int_0^{2\pi} d\theta = \frac{25}{4\pi}\cos(7(t_1 - t_2))\, 2\pi = \boxed{\frac{25}{2}\cos(7(t_1 - t_2))}.$$

[1] This identity could be derived easily via the Euler's identity:

$$\begin{aligned}\cos(a) \times \cos(b) &= \frac{e^{ja} + e^{-ja}}{2} \times \frac{e^{jb} + e^{-jb}}{2} = \frac{1}{4}\left(e^{ja}e^{jb} + e^{-ja}e^{jb} + e^{ja}e^{-jb} + e^{-ja}e^{-jb}\right) \\ &= \frac{1}{2}\left(\frac{e^{ja}e^{jb} + e^{-ja}e^{-jb}}{2} + \frac{e^{-ja}e^{jb} + e^{ja}e^{-jb}}{2}\right) \\ &= \frac{1}{2}\left(\cos(a + b) + \cos(a - b)\right).\end{aligned}$$

Extra Question

Here is an optional question for those who want more practice.

Problem 5. Let X be a uniform random variable on the interval $[0,1]$. Set

$$A = \left[0, \frac{1}{2}\right), \quad B = \left[0, \frac{1}{4}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right), \quad \text{and } C = \left[0, \frac{1}{8}\right) \cup \left[\frac{1}{4}, \frac{3}{8}\right) \cup \left[\frac{1}{2}, \frac{5}{8}\right) \cup \left[\frac{3}{4}, \frac{7}{8}\right).$$

Are the events $[X \in A], [X \in B]$, and $[X \in C]$ independent?

Solution: Note that

$$P[X \in A] = \int_0^{\frac{1}{2}} dx = \frac{1}{2},$$

$$P[X \in B] = \int_0^{\frac{1}{4}} dx + \int_{\frac{1}{2}}^{\frac{3}{4}} dx = \frac{1}{2}, \text{ and}$$

$$P[X \in C] = \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{4}}^{\frac{3}{8}} dx + \int_{\frac{1}{2}}^{\frac{5}{8}} dx + \int_{\frac{3}{4}}^{\frac{7}{8}} dx = \frac{1}{2}.$$

Now, for pairs of events, we have

$$P([X \in A] \cap [X \in B]) = \int_0^{\frac{1}{4}} dx = \frac{1}{4} = P[X \in A] \times P[X \in B], \tag{12.1}$$

$$P([X \in A] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{4}}^{\frac{3}{8}} dx = \frac{1}{4} = P[X \in A] \times P[X \in C], \text{ and} \tag{12.2}$$

$$P([X \in B] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{2}}^{\frac{5}{8}} dx = \frac{1}{4} = P[X \in B] \times P[X \in C]. \tag{12.3}$$

Finally,

$$P([X \in A] \cap [X \in B] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx = \frac{1}{8} = P[X \in A] P[X \in B] P[X \in C]. \tag{12.4}$$

From (12.1), (12.2), (12.3) and (12.4), we can conclude that the events $[X \in A], [X \in B]$, and $[X \in C]$ are $\boxed{\text{independent}}$.