

Y&G Q3.2.1

Wednesday, October 03, 2012 3:50 PM

In this question, you are given a pdf whose expression has an unknown constant c .

(a) To find the constant c , recall that any pdf should integrate to 1.

In this problem,

$$\int_{-\infty}^{\infty} f_x(x) dx = \int_0^2 c x dx = c \int_0^2 x dx = c \frac{x^2}{2} \Big|_0^2 = 2c$$

This should = 1.

Therefore, $c = \frac{1}{2}$.

(b) $P[0 \leq X \leq 1] = \int_0^1 f_x(x) dx = \int_0^1 \frac{1}{2} x dx = \frac{1}{2} \frac{x^2}{2} \Big|_0^1 = \frac{1}{4}$.

(c) $P[-\frac{1}{2} \leq X \leq \frac{1}{2}] = \int_{-1/2}^{1/2} f_x(x) dx = \int_0^{1/2} \frac{1}{2} x dx = \frac{1}{2} \frac{x^2}{2} \Big|_0^{1/2} = \frac{1}{16}$.

$f_x(x) = 0$ on $[-\frac{1}{2}, 0)$

(d) For $x < 0$, because $f_x(t) = 0$ for $t < 0$, $F_x(x) = \int_{-\infty}^x f_x(t) dt = 0$

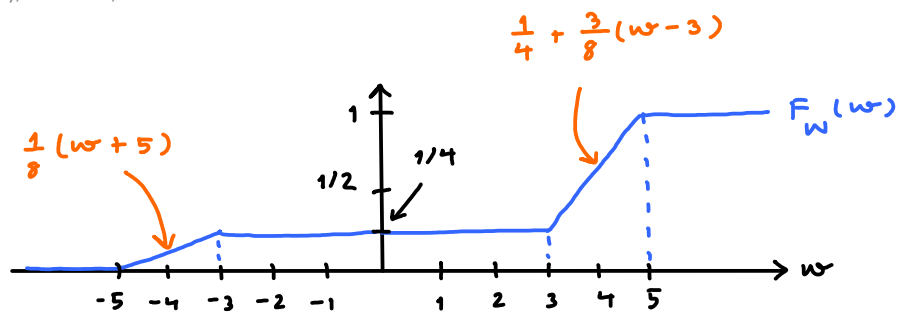
For $0 \leq x \leq 2$, $f_x(t) = \frac{t}{2}$ and $F_x(x) = \int_{-\infty}^x f_x(t) dt = \int_0^x \frac{t}{2} dt = \frac{t^2}{4} \Big|_0^x = \frac{x^2}{4}$.

At $x = 2$, $F_x(2) = 1$.

For $x > 2$, $f_x(t) = 0$. Therefore, $F_x(x) = \int_{-\infty}^x f_x(t) dt = \underbrace{\int_{-\infty}^2 f_x(t) dt}_{F_x(2) = 1} + \underbrace{\int_2^x f_x(t) dt}_0 = 1$.

Combining the three results above, we have

$$F_x(x) = \begin{cases} 0, & x < 0, \\ x^2/4, & 0 \leq x \leq 2, \\ 1, & \text{otherwise.} \end{cases}$$



(a) From the plot above, we see that $F_W(w)$ is a continuous function.
Because its cdf is continuous, we conclude that W is a continuous RV.

$$(b) P[W \leq 4] = F_W(4) = \frac{1}{4} + \frac{3}{8}(4-3) = \frac{1}{4} + \frac{3}{8} = \frac{5}{8} \approx 0.625$$

↑
by definition of cdf

$$(c) P[-2 < W \leq 2] = F_W(2) - F_W(-2) = \frac{1}{4} - \frac{1}{4} = 0$$

↑
For continuous RV, $P[a \leq X \leq b] = F_X(b) - F_X(a)$

$$(d) P[W > 0] = 1 - P[W \leq 0] = 1 - F_W(0) = 1 - \frac{1}{4} = \frac{3}{4}$$

↑
 $P(A) = 1 - P(A^c)$

(e) $P[W \leq a] = F_W(a)$. From the plot above, we know that to have $F_W(a) = \frac{1}{2}$, the value of a must be in the interval $(3, 5)$.
In this interval, $F_W(a) = \frac{1}{4} + \frac{3}{8}(a-3)$.

So, we solve for "a" that satisfies

$$\frac{1}{4} + \frac{3}{8}(a-3) = \frac{1}{2} \Rightarrow a = \frac{11}{3} \approx 3.67$$

Remark: It is possible to solve this problem by finding the pdf first.

(I ask you to derive the pdf anyway in Q 3.2.3.)

However, you should also make sure that you know how to evaluate the probabilities above directly from the cdf.

Y&G Q3.2.3

Wednesday, October 03, 2012 4:18 PM

Given a cdf, we can find the pdf by taking derivative.

As discussed in class, for the location(s) where derivative does not exist, we can choose to define the pdf to be any convenient value.

In this question, the cdf is given in the form of expressions on several intervals. It is then easy to find its derivative inside each of the intervals:

$$f_w(w) = \frac{d}{dw} F_w(w) = \begin{cases} 0, & w < -5, \\ 1/8, & -5 < w < -3, \\ 0, & -3 < w < 3, \\ 3/8, & 3 < w < 5, \\ 0, & 5 < w. \end{cases}$$

It should be clear from the plot of cdf in Q 3.1.3 that the derivative does not exist at $w = -5, -3, 3, 5$. We choose to assign $f_w(w) = 0$ at these points.

$$f_w(w) = \begin{cases} 1/8, & -5 < w < -3 \\ 3/8, & 3 < w < 5 \\ 0, & \text{otherwise} \end{cases}$$