

## HW Solution 7 — Due: Oct 28, 9:19 AM (in tutorial session)

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**Instructions**

- (a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.
- (b) It is important that you try to solve all problems. (5 pt)  
The extra questions at the end are optional.
- (c) Late submission will be heavily penalized.

**Problem 1.** [F2013/1] For each of the following random variables, find  $P[1 < X \leq 2]$ .

- (a)  $X \sim \text{Binomial}(3, 1/3)$
- (b)  $X \sim \text{Poisson}(3)$

**Solution:**

- (a) Because  $X \sim \text{Binomial}(3, 1/3)$ , we know that  $X$  can only take the values 0, 1, 2, 3. Only the value 2 satisfies the condition given. Therefore,  $P[1 < X \leq 2] = P[X = 2] = p_X(2)$ . Recall that the pmf for the binomial random variable is

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for  $x = 0, 1, 2, 3, \dots, n$ . Here, it is given that  $n = 3$  and  $p = 1/3$ . Therefore,

$$p_X(2) = \binom{3}{2} \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{3}\right)^{3-2} = 3 \times \frac{1}{9} \times \frac{2}{3} = \boxed{\frac{2}{9}}.$$

- (b) Because  $X \sim \text{Poisson}(3)$ , we know that  $X$  can take the values 0, 1, 2, 3,  $\dots$ . As in the previous part, only the value 2 satisfies the condition given. Therefore,  $P[1 < X \leq 2] = P[X = 2] = p_X(2)$ . Recall that the pmf for the Poisson random variable is

$$p_X(x) = e^{-\alpha} \frac{\alpha^x}{x!}$$

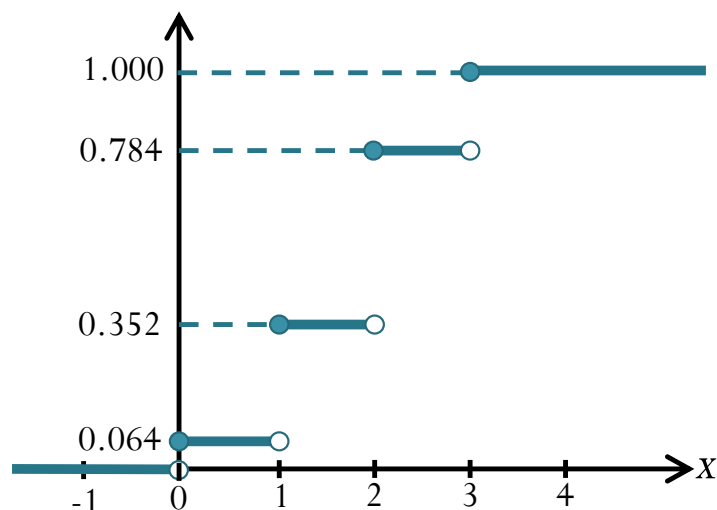


Figure 7.1: CDF of X for Problem 2

for  $x = 0, 1, 2, 3, \dots$ . Here, it is given that  $\alpha = 3$ . Therefore,

$$p_X(2) = e^{-3} \frac{3^2}{2!} = \boxed{\frac{9}{2} e^{-3} \approx 0.2240}.$$

**Problem 2.** [M2011/1] The cdf of a random variable  $X$  is plotted in Figure 7.1.

- Find the pmf  $p_X(x)$ .
- Find the family to which  $X$  belongs. (Uniform, Bernoulli, Binomial, Geometric, Poisson, etc.)

**Solution:**

- For discrete random variable,  $P[X = x]$  is the jump size at  $x$  on the cdf plot. In this problem, there are four jumps at 0, 1, 2, 3.
  - $P[X = 0] =$  the jump size at 0  $= 0.064 = \frac{64}{1000} = (4/10)^3 = (2/5)^3$ .
  - $P[X = 1] =$  the jump size at 1  $= 0.352 - 0.064 = 0.288$ .
  - $P[X = 2] =$  the jump size at 2  $= 0.784 - 0.352 = 0.432$ .
  - $P[X = 3] =$  the jump size at 3  $= 1 - 0.784 = 0.216 = (6/10)^3$ .

In conclusion,

$$p_X(x) = \begin{cases} 0.064, & x = 0, \\ 0.288, & x = 1, \\ 0.432, & x = 2, \\ 0.216, & x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

- (b) Among all the pmf that we discussed in class, only one can have support =  $\{0, 1, 2, 3\}$  with unequal probabilities. This is the binomial pmf. To check that it really is Binomial, recall that the pmf for binomial  $X$  is given by  $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$  for  $x = 0, 1, 2, \dots, n$ . Here,  $n = 3$ . Furthermore, observe that  $p_X(0) = (1-p)^n$ . By comparing  $p_X(0)$  with what we had in part (a), we have  $1-p = 2/5$  or  $p = 3/5$ . For  $x = 1, 2, 3$ , plugging in  $p = 3/5$  and  $n = 3$  in to  $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$  gives the same values as what we had in part (a). So,  $X$  is a binomial RV.

**Problem 3.** Arrivals of customers at the local supermarket are modeled by a Poisson process with a rate of  $\lambda = 2$  customers per minute. Let  $M$  be the number of customers arriving between 9:00 and 9:05. What is the probability that  $M < 2$ ?

**Solution:** Here, we are given that  $M \sim \mathcal{P}(\alpha)$  where  $\alpha = \lambda T = 2 \times 5 = 10$ . Recall that, for  $M \sim \mathcal{P}(\alpha)$ , we have

$$P[M = m] = \begin{cases} e^{-\alpha} \frac{\alpha^m}{m!}, & m \in \{0, 1, 2, 3, \dots\} \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} P[M < 2] &= P[M = 0] + P[M = 1] = e^{-\alpha} \frac{\alpha^0}{0!} + e^{-\alpha} \frac{\alpha^1}{1!} \\ &= e^{-\alpha} (1 + \alpha) = e^{-10} (1 + 10) = 11e^{-10} \approx 5 \times 10^{-4}. \end{aligned}$$

**Problem 4.** When  $n$  is large, binomial distribution Binomial( $n, p$ ) becomes difficult to compute directly because of the need to calculate factorial terms. In this question, we will consider an approximation when the value of  $p$  is close to 0. In such case, the binomial can be approximated<sup>1</sup> by the Poisson distribution with parameter  $\alpha = np$ . For this approximation to work, we will see in this exercise that  $n$  does not have to be very large and  $p$  does not need to be very small.

<sup>1</sup>More specifically, suppose  $X_n$  has a binomial distribution with parameters  $n$  and  $p_n$ . If  $p_n \rightarrow 0$  and  $np_n \rightarrow \alpha$  as  $n \rightarrow \infty$ , then

$$P[X_n = k] \rightarrow e^{-\alpha} \frac{\alpha^k}{k!}.$$

- (a) Let  $X \sim \text{Binomial}(12, 1/36)$ . (For example, roll two dice 12 times and let  $X$  be the number of times a double 6 appears.) Evaluate  $p_X(x)$  for  $x = 0, 1, 2$ .
- (b) Compare your answers part (a) with its Poisson approximation.
- (c) Compare MATLAB plots of  $p_X(x)$  in part (a) and the pmf of  $\mathcal{P}(np)$ .

**Solution:**

- (a) For Binomial( $n, p$ ) random variable,

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, 2, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we are given that  $n = 12$  and  $p = \frac{1}{36}$ . Plugging in  $x = 0, 1, 2$ , we get 0.7132, 0.2445, 0.0384, respectively

- (b) A Poisson random variable with parameter  $\alpha = np$  can approximate a Binomial( $n, p$ ) random variable when  $n$  is large and  $p$  is small. Here, with  $n = 12$  and  $p = \frac{1}{36}$ , we have  $\alpha = 12 \times \frac{1}{36} = \frac{1}{3}$ . The Poisson pmf at  $x = 0, 1, 2$  is given by  $e^{-\alpha} \frac{\alpha^x}{x!} = e^{-1/3} \frac{(1/3)^x}{x!}$ . Plugging in  $x = 0, 1, 2$  gives 0.7165, 0.2388, 0.0398, respectively.
- (c) See Figure 7.2. Note how close they are!

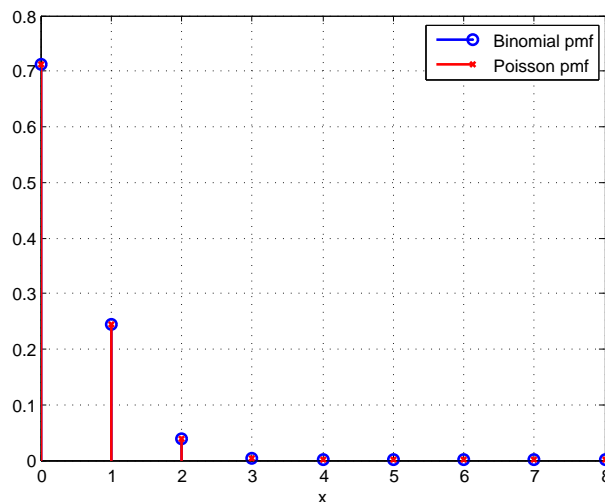


Figure 7.2: Poisson Approximation

**Problem 5.** You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson pmf. (For simplicity, exclude birthdays on February 29.) [Bertsekas and Tsitsiklis, 2008, Q2.2.2]

**Solution:** Let  $N$  be the number of guests that has the same birthday as you. We may think of the comparison of your birthday with each of the guests as a Bernoulli trial. Here, there are 500 guests and therefore we are considering  $n = 500$  trials. For each trial, the (success) probability that you have the same birthday as the corresponding guest is  $p = \frac{1}{365}$ . Then, this  $N \sim \text{Binomial}(n, p)$ .

(a) Binomial:  $P[N = 1] = np^1(1 - p)^{n-1} \approx 0.348$ .

(b) Poisson:  $P[N = 1] = e^{-np} \frac{(np)^1}{1!} \approx 0.348$ .

## Extra Questions

Here are some optional questions for those who want more practice.

**Problem 6.** A sample of a radioactive material emits particles at a rate of 0.7 per second. Assuming that these are emitted in accordance with a Poisson distribution, find the probability that in one second

- (a) exactly one is emitted,
- (b) more than three are emitted,
- (c) between one and four (inclusive) are emitted

[Applebaum, 2008, Q5.27].

**Solution:** Let  $X$  be the number of particles emitted during the one second under consideration. Then  $X \sim \mathcal{P}(\alpha)$  where  $\alpha = \lambda T = 0.7 \times 1 = 0.7$ .

$$(a) P[X = 1] = e^{-\alpha} \frac{\alpha^1}{1!} = \alpha e^{-\alpha} = 0.7e^{-0.7} \approx 0.3477.$$

$$(b) P[X > 3] = 1 - P[X \leq 3] = 1 - \sum_{k=0}^3 e^{-0.7} \frac{0.7^k}{k!} \approx 0.0058.$$

$$(c) P[1 \leq X \leq 4] = \sum_{k=1}^4 e^{-0.7} \frac{0.7^k}{k!} \approx 0.5026.$$

**Problem 7** (M2011/1). You are given an unfair coin with probability of obtaining a head equal to  $1/3,000,000,000$ . You toss this coin  $6,000,000,000$  times. Let  $A$  be the event that you get “tails for all the tosses”. Let  $B$  be the event that you get “heads for all the tosses”.

- (a) Approximate  $P(A)$ .
- (b) Approximate  $P(A \cup B)$ .

**Solution:** Let  $N$  be the number of heads among the  $n$  tosses. Then,  $N \sim \mathcal{B}(n, p)$ . Here, we have small  $p = 1/3 \times 10^9$  and large  $n = 6 \times 10^9$ . So, we can apply Poisson approximation. In other words,  $\mathcal{B}(n, p)$  is well-approximated by  $\mathcal{P}(\alpha)$  where  $\alpha = np = 2$ .

$$(a) P(A) = P[N = 0] = e^{-2} \frac{2^0}{0!} = \frac{1}{e^2} \approx 0.1353.$$

$$(b) P(A \cup B) = P[N = 0] + P[N = n] = e^{-2} \frac{2^0}{0!} + e^{-2} \frac{2^{6 \times 10^9}}{(6 \times 10^9)!}.$$

The second term is extremely small compared to the first one. Hence,  $P(A \cup B)$  is approximately the same as  $P(A)$ .