ECS 315: Probability and Random Processes **2015/1**

HW Solution 5 — Due: Sep 23, 9:19 AM (in tutorial session)

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Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)
The extra questions at the end are optional.

(c) Late submission will be heavily penalized.

Problem 1. In an experiment, A, B, C, and D are events with probabilities $P(A \cup B) = \frac{5}{8}$, $P(A) = \frac{3}{8}$, $P(C \cap D) = \frac{1}{3}$, and $P(C) = \frac{1}{2}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find

(i) $P(A \cap B)$

(ii) $P(B)$

(iii) $P(A \cap B^c)$

(iv) $P(A \cup B^c)$

(b) Are A and B independent?

(c) Find

(i) $P(D)$

(ii) $P(C \cap D^c)$

(iii) $P(C^c \cap D^c)$

(iv) $P(C|D)$

(v) $P(C \cup D)$

(vi) $P(C \cup D^c)$

(d) Are C and D^c independent?

Solution:

(a)

(i) Because $A \perp B$, we have $A \cap B = \emptyset$ and hence $P(A \cap B) = \boxed{0}$.

(ii) Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, $P(B) = P(A \cup B) - P(A) + P(A \cap B) = 5/8 - 3/8 + 0 = 2/8 = \boxed{1/4}$.

(iii) $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) = \boxed{3/8}$.

(iv) Start with $P(A \cup B^c) = 1 - P(A^c \cap B)$. Now, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) = 1/4$. Hence, $P(A \cup B^c) = 1 - 1/4 = \boxed{3/4}$.

(b) Events A and B are not independent because $P(A \cap B) \neq P(A)P(B)$.

(c)

(i) Because $C \perp\!\!\!\perp D$, we have $P(C \cap D) = P(C)P(D)$. Hence, $P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1/3}{1/2} = \boxed{2/3}$.

(ii) **Method 1**: $P(C \cap D^c) = P(C) - P(C \cap D) = 1/2 - 1/3 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C \perp\!\!\!\perp D^c$. Hence, $P(C \cap D^c) = P(C)P(D^c) = \frac{1}{2}\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iii) **Method 1**: First, we find $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = 5/6$. Hence, $P(C^c \cap D^c) = 1 - P(C \cup D) = 1 - 5/6 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C^c \perp\!\!\!\perp D^c$. Hence, $P(C^c \cap D^c) = P(C^c)P(D^c) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iv) Because $C \perp\!\!\!\perp D$, we have $P(C|D) = P(C) = \boxed{1/2}$.

(v) In part (iii), we already found $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = \boxed{5/6}$.

(vi) **Method 1**: $P(C \cup D^c) = 1 - P(C^c \cap D) = 1 - P(C^c)P(D) = 1 - \frac{1}{2}\frac{2}{3} = \boxed{2/3}$.
Note that we use the fact that $C^c \perp\!\!\!\perp D$ to get the second equality.
Method 2: Alternatively, $P(C \cup D^c) = P(C) + P(D^c) - P(C \cap D^C)$. From (i), we have $P(D) = 2/3$. Hence, $P(D^c) = 1 - 2/3 = 1/3$. From (ii), we have $P(C \cap D^C) = 1/6$. Therefore, $P(C \cup D^c) = 1/2 + 1/3 - 1/6 = 2/3$.

(d) Yes. We know that if $C \perp\!\!\!\perp D$, then $C \perp\!\!\!\perp D^c$.

Problem 2. In this question, each experiment has equiprobable outcomes.

(a) Let $\Omega = \{1,2,3,4\}$, $A_1 = \{1,2\}$, $A_2 = \{1,3\}$, $A_3 = \{2,3\}$.

  (i) Determine whether $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.

  (ii) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

  (iii) Are A_1, A_2, and A_3 independent?

(b) Let $\Omega = \{1,2,3,4,5,6\}$, $A_1 = \{1,2,3,4\}$, $A_2 = A_3 = \{4,5,6\}$.

  (i) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

  (ii) Check whether $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.

  (iii) Are A_1, A_2, and A_3 independent?

Solution:

(a) We have $P(A_i) = \frac{1}{2}$ and $P(A_i \cap A_j) = \frac{1}{4}$.

  (i) $P(A_i \cap A_j) = P(A_i)P(A_j)$ for any $i \neq j$.

  (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$. Hence, $P(A_1 \cap A_2 \cap A_3) = 0$, which is *not* the same as $P(A_1)P(A_2)P(A_3)$.

  (iii) No.

  Remark: This counter-example shows that pairwise independence does not imply independence.

(b) We have $P(A_1) = \frac{4}{6} = \frac{2}{3}$ and $P(A_2) = P(A_3) = \frac{3}{6} = \frac{1}{2}$.

  (i) $A_1 \cap A_2 \cap A_3 = \{4\}$. Hence, $P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}$.
  $P(A_1)P(A_2)P(A_3) = \frac{2}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{6}$.
  Hence, $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

  (ii) $P(A_2 \cap A_3) = P(A_2) = \frac{1}{2} \neq P(A_2)P(A_3)$
  $P(A_1 \cap A_2) = p(4) = \frac{1}{6} \neq P(A_1)P(A_2)$
  $P(A_1 \cap A_3) = p(4) = \frac{1}{6} \neq P(A_1)P(A_3)$
  Hence, $P(A_i \cap A_j) \neq P(A_i)P(A_j)$ for all $i \neq j$.

  (iii) No.

  Remark: This counter-example shows that one product condition does not imply independence.

Problem 3. Series Circuit: The circuit in Figure 5.1 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-32]

Figure 5.1: Circuit for Problem 3

Solution: Let L and R denote the events that the left and right devices operate, respectively. For a path to exist, both need to operate. Therefore, the probability that the circuit operates is $P(L \cap R)$.

We are told that $L^c \perp\!\!\!\perp R^c$. This is equivalent to $L \perp\!\!\!\perp R$. By their independence,

$$P(L \cap R) = P(L)P(R) = 0.8 \times 0.9 = \boxed{0.72}.$$

Problem 4 (Majority Voting in Digital Communication)**.** A certain binary communication system has a bit-error rate of 0.1; i.e., in transmitting a single bit, the probability of receiving the bit in error is 0.1. To transmit messages, a three-bit repetition code is used. In other words, to send the message 1, a "codeword" 111 is transmitted, and to send the message 0, a "codeword" 000 is transmitted. At the receiver, if two or more 1s are received, the decoder decides that message 1 was sent; otherwise, i.e., if two or more zeros are received, it decides that message 0 was sent.

Assuming bit errors occur independently, find the probability that the decoder puts out the wrong message.

[Gubner, 2006, Q2.62]

Solution: Let $p = 0.1$ be the bit error rate. Let $\mathcal{E}$ be the error event. (This is the event that the decoded bit value is not the same as the transmitted bit value.) Because majority voting is used, event $\mathcal{E}$ occurs if and only if there are at least two bit errors. Therefore

$$P(\mathcal{E}) = \binom{3}{2} p^2(1-p) + \binom{3}{3} p^3 = p^2(3-2p).$$

When $p = 0.1$, we have $P(\mathcal{E}) \approx \boxed{0.028}$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases. A specific design is randomly generated by the Web server when you visit the site. Let A denote the event that the design color is red and let B denote the event that the font size is not the smallest one.

(a) Use classical probability to evaluate $P(A)$, $P(B)$ and $P(A \cap B)$. Show that the two events A and B are independent by checking whether $P(A \cap B) = P(A)P(B)$.

(b) Using the values of $P(A)$ and $P(B)$ from the previous part and the fact that $A \perp\!\!\!\perp B$, calculate the following probabilities.

(i) $P(A \cup B)$

(ii) $P(A \cup B^c)$

(iii) $P(A^c \cup B^c)$

[Montgomery and Runger, 2010, Q2-84]

Solution:

(a) By multiplication rule, there are

$$|\Omega| = 4 \times 3 \times 5 \times 3 \times 5 \tag{5.1}$$

possible designs. The number of designs whose color is red is given by

$$|A| = 1 \times 3 \times 5 \times 3 \times 5.$$

Note that the "4" in (**??**) is replaced by "1" because we only consider one color (red). Therefore,

$$P(A) = \frac{1 \times 3 \times 5 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{1}{4}}.$$

Similarly, $|B| = 4 \times 3 \times 4 \times 3 \times 5$ where the "5" in the middle of (**??**) is replaced by "4" because we can't use the smallest font size. Therefore,

$$P(B) = \frac{4 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{4}{5}}.$$

For the event $A \cap B$, we replace "4" in (**??**) by "1" because we need red color and we replace "5" in the middle of (**??**) by "4" because we can't use the smallest font size. This gives

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{1 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \frac{1 \times 4}{4 \times 5} = \boxed{\frac{1}{5}} = 0.2.$$

Because $P(A \cap B) = P(A)P(B)$, the events A and B are independent.

(b)

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{4}{5} - \frac{1}{5} = \boxed{\frac{17}{20} = 0.85.}$

(ii) **Method 1**: $P(A \cup B^c) = 1 - P\left((A \cup B^c)^c\right) = 1 - P(A^c \cap B)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$. Hence, $P(A^c \cup B^c) = 1 - P(A^c)P(B) = 1 - \frac{3}{4}\frac{4}{5} = \frac{2}{5} = \boxed{0.4.}$

Method 2: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = B^c \cup (A \cap B)$. Therefore,

$$P(A \cup B^c) = P(B^c) + P(A \cap B) = 1 - P(B) + P(A)P(B) = 1 - \frac{4}{5} + \frac{1}{4}\frac{4}{5} = \frac{2}{5}.$$

Method 3: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = A \cup (A^c \cap B^c)$. Therefore, $P(A \cup B^c) = P(A) + P(A^c \cap B^c)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B^c$. Hence,

$$P(A \cup B^c) = P(A) + P(A^c)P(B^c) = P(A) + (1 - P(A))(1 - P(B)) = \frac{1}{4} + \frac{3}{4}\frac{1}{5} = \frac{2}{5}.$$

(iii) **Method 1**: $P(A^c \cup B^c) = 1 - P\left((A^c \cup B^c)^c\right) = 1 - P(A \cap B) = 1 - 0.2 = \boxed{0.8.}$

Method 2: From the Venn diagram, note that $A^c \cup B^c$ can be expressed as a disjoint union: $A^c \cup B^c = (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c)$. Therefore,

$$P\left(A^c \cup B^c\right) = P\left(A^c \cap B\right) + P\left(A \cap B^c\right) + P\left(A^c \cap B^c\right).$$

Now, because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$, $A \perp\!\!\!\perp B^c$, and $A^c \perp\!\!\!\perp B^c$. Hence,

$$\begin{aligned} P\left(A^c \cup B^c\right) &= P\left(A^c\right)P\left(B\right) + P\left(A\right)P\left(B^c\right) + P\left(A^c\right)P\left(B^c\right) \\ &= \left(1 - P\left(A\right)\right)P\left(B\right) + P\left(A\right)\left(1 - P\left(B\right)\right) + \left(1 - P\left(A\right)\right)\left(1 - P\left(B\right)\right) \\ &= \frac{3}{4} \times \frac{4}{5} + \frac{1}{4} \times \frac{1}{5} + \frac{3}{4} \times \frac{1}{5} = \frac{16}{20} = \frac{4}{5} \end{aligned}$$

Problem 6. Show that if A and B are independent events, then so are A and B^c, A^c and B, and A^c and B^c.

Solution: To show that two events C_1 and C_2 are independent, we need to show that $P(C_1 \cap C_2) = P(C_1)P(C_2)$.

(a) Note that

$$P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B).$$

Because $A \perp\!\!\!\perp B$, the last term can be factored in to $P(A)P(B)$ and hence

$$P(A \cap B^c) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$$

(b) By interchanging the role of A and B in the previous part, we have

$$P(A^c \cap B) = P(B \cap A^c) = P(B)\,P(A^c)\,.$$

(c) From set theory, we know that $A^c \cap B^c = (A \cup B)^c$. Therefore,

$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)\,,$$

where, for the last equality, we use

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

which is discussed in class.

Because $A \perp\!\!\!\perp B$, we have

$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))\,(1 - P(B)) \\ &= P(A^c)P(B^c). \end{aligned}$$

Remark: By interchanging the roles of A and A^c and/or B and B^c, it follows that if any one of the four pairs is independent, then so are the other three. [Gubner, 2006, p.31]

Problem 7. Anne and Betty go fishing. Find the conditional probability that Anne catches no fish given that at least one of them catches no fish. Assume they catch fish independently and that each has probability $0 < p < 1$ of catching no fish. [Gubner, 2006, Q2.62]

Hint: Let A be the event that Anne catches no fish and B be the event that Betty catches no fish. Observe that the question asks you to evaluate $P(A|(A \cup B))$.

Solution: From the question, we know that A and B are independent. The event "at least one of the two women catches nothing" can be represented by $A \cup B$. So we have

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B) - P(A)\,P(B)} = \frac{p}{2p - p^2} = \boxed{\frac{1}{2 - p}}.$$

Problem 8. The circuit in Figure 5.2 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-34]

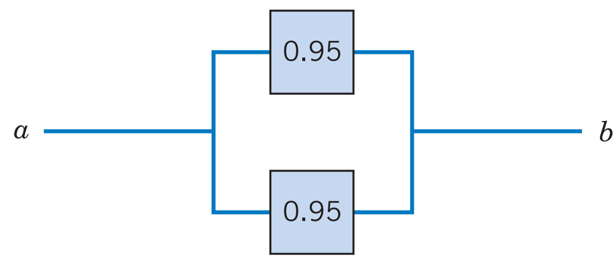

Figure 5.2: Circuit for Problem 8

Solution: Let T and B denote the events that the top and bottom devices operate, respectively. There is a path if at least one device operates. Therefore, the probability that the circuit operates is $P(T \cup B)$. Note that

$$P\left(T \cup B\right) = 1 - P\left(\left(T \cup B\right)^c\right) = 1 - P\left(T^c \cap B^c\right).$$

We are told that $T^c \perp\!\!\!\perp B^c$. By their independence,

$$P(T^c \cap B^c) = P(T^c)P(B^c) = (1 - 0.95) \times (1 - 0.95) = 0.05^2 = 0.0025.$$

Therefore,

$$P\left(T \cup B\right) = 1 - P\left(T^c \cap B^c\right) = 1 - 0.0025 = \boxed{0.9975}.$$

Problem 9. The circuit in Figure 5.3 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-35]

Solution: The solution can be obtained from a partition of the graph into three columns. Let L denote the event that there is a path of functional devices only through the three units on the left. From the independence and based upon Problem 8,

$$P(L) = 1 - (1 - 0.9)^3 = 1 - 0.1^3 = 0.999.$$

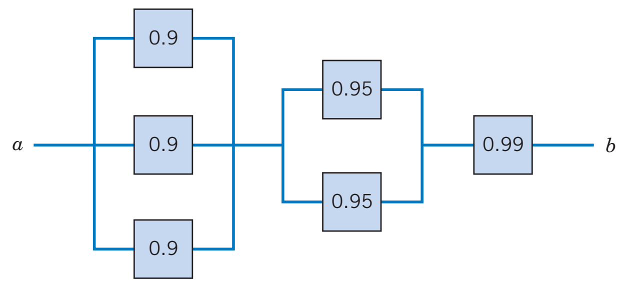

Figure 5.3: Circuit for Problem 9

Similarly, let M denote the event that there is a path of functional devices only through the two units in the middle. Then,

$$P(M) = 1 - (1 - 0.95)^2 = 1 - 0.05^2 = 1 - 0.0025 = 0.9975.$$

Finally, the probability that there is a path of functional devices only through the one unit on the right is simply the probability that the device functions, namely, 0.99.

Therefore, with the independence assumption used again, along with similar reasoning to the solution of Problem 3, the solution is

$$0.999 \times 0.9975 \times 0.99 = 0.986537475 \approx \boxed{0.987}.$$

Problem 10. ***Binomial theorem***: For any positive integer n, we know that

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{5.2}$$

(a) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

(b) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

(c) Use the binomial theorem (5.1) to evaluate $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

(d) Use the binomial theorem (5.1) to simplify the following sums

(i) $\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r (1 - x)^{n-r}$

(ii) $\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r (1 - x)^{n-r}$

(e) If we differentiate (5.1) with respect to x and then multiply by x, we have

$$\sum_{r=0}^{n} r\binom{n}{r}x^r y^{n-r} = nx(x+y)^{n-1}.$$

Use similar technique to simplify the sum $\sum_{r=0}^{n} r^2\binom{n}{r}x^r y^{n-r}$.

Solution:

(a) $\binom{25}{12} = \boxed{5,200,300}$.

(b) $\binom{25}{12}2^{12}(-3)^{13} = -\frac{25!}{12!13!}2^{12}3^{13} = \boxed{-33959763545702400}$.

(c) From (5.1), set $x = -1$ and $y = 1$, then we have $\sum\limits_{k=0}^{n}(-1)^k\binom{n}{k} = (-1+1)^n = \boxed{0}$.

(d) To deal with the sum involving only the even terms (or only the odd terms), we first use (5.1) to expand $(x+y)^n$ and $(x+(-y))^n$. When we add the expanded results, only the even terms in the sum are left. Similarly, when we find the difference between the two expanded results, only the the odd terms are left. More specifically,

$$\sum_{\substack{r=0\\ r \text{ even}}}^{n}\binom{n}{r}x^r y^{n-r} = \frac{1}{2}\left((x+y)^n + (y-x)^n\right), \text{ and}$$

$$\sum_{\substack{r=0\\ r \text{ odd}}}^{n}\binom{n}{r}x^r y^{n-r} = \frac{1}{2}\left((x+y)^n - (y-x)^n\right).$$

If $x + y = 1$, then

$$\sum_{\substack{r=0\\ r \text{ even}}}^{n}\binom{n}{r}x^r y^{n-r} = \boxed{\frac{1}{2}\left(1+(1-2x)^n\right)}, \text{ and} \tag{5.3a}$$

$$\sum_{\substack{r=0\\ r \text{ odd}}}^{n}\binom{n}{r}x^r y^{n-r} = \boxed{\frac{1}{2}\left(1-(1-2x)^n\right)}. \tag{5.3b}$$

(e) $\sum_{r=0}^{n} r^2\binom{n}{r}x^r y^{n-r} = \boxed{nx\left(x(n-1)(x+y)^{n-2} + (x+y)^{n-1}\right)}$.

Problem 11. (**Classical Probability and Combinatorics**) Suppose n integers are chosen with replacement (that is, the same integer could be chosen repeatedly) at random from $\{1, 2, 3, \ldots, N\}$. Calculate the probability that the chosen numbers arise according to some non-decreasing sequence.

Solution: There are N^n possible sequences. (This is ordered sampling with replacement.) To find the probability, we need to count the number of non-decreasing sequences among these N^n possible sequences. It takes some thought to realize that this is exactly the counting problem that we called "unordered sampling with replacement". In which case, we can conclude that the probability is $\boxed{\frac{\binom{n+N-1}{n}}{N^n}}$. The "with replacement" part should be clear from the question statement. The "unordered" part needs some more thought.

To see this, let's look back at how we turn the "ordered sampling *without replacement*" into "unordered sampling *without replacement*". Recall that there are $(N)_n$ distinct samples for "ordered sampling without replacement". When we switch to the "unordered" case, we see that many of the original samples from the "ordered sampling without replacement" are regarded as the same in the "unordered" case. In fact, we can form "groups" of samples whose members are regarded as the same in the "unordered" case. We can then count the number of groups. In class, we found that the size of any individual group can be calculated easily from permuting the elements in a sample and hence there are $n!$ members in each group. This leads us to conclude that there are $(N)_n/n! = \binom{N}{n}$ groups.

We are in a similar situation when we want to turn the "ordered sampling *with replacement*" into "unordered sampling *with replacement*". We first start with N^n distinct samples from "ordered sampling with replacement". Now, we again separate these samples into groups. Let's consider an example where $n = 3$. Then sequences "1 1 2", "1 2 1", and "2 1 1" are put together in the same group in the "unordered" case. Note that the size of this group is 3. The sequences "1 2 3", "1 3 2", "2 1 3", "2 3 1", "3 1 2", and "3 2 1" are in another group. Note that the size of this group is 6. Therefore, the group sizes are not the same and hence we can not find the number of groups by $N^n/(\text{group size})$ as in the sampling *without replacement* discussed in the previous paragraph. To count the number of groups, we look at the sequences from another perspective. We see that the "unordered" case, the only information the characterizes each group is "how many of each number there are". This is why we can match the number of groups to the number of nonnegative-integer solutions to the equation $x_1 + x_2 + \cdots + x_N = n$ as discussed in class. Finally, note that for each group, we have only one possible nondecreasing sequence. So, the number of possible nondecreasing sequence is the same as the number of groups.

If you think about the explanation above, you may realize that, by requiring the "order" on the sequence, the counting problem become "unordered sampling".

Here, we present two direct methods that leads to the same answer.

Method 1: Because the sequence is non-decreasing, the number of times that each of the integers $\{1, 2, \dots, N\}$ shows up in the sequence is the only information that characterizes each sequence. Let x_i be the number of times that number i shows up in the sequence. The number of sequences is then the same as the number of solution to the equation $x_1+x_2+\cdots+x_N = n$ where the x_i are all non-negative integers. We have seen in class that the number of solutions is $\binom{n+N-1}{n}$.

Method 2: [DasGupta, 2010, Example 1.14, p. 12] Consider the following construction of such non-decreasing sequence. Start with n stars and $N-1$ bars. There are $\binom{n+N-1}{n}$ arrangements of these. For example, when $N = 5$ and $n = 2$, one arrangement is $| * || * |$. Now, add spaces between these bars and stars including before the first one and after the last one. For our earlier example, we have $\_|\_*\_|\_|\_*\_|\_$. Now, put number 1 in the leftmost space. After this position, the next space holds the same value as the previous on if you pass a $*$. On the other hand, if you pass a $|$ then the value increases by 1. Note that because there are $N-1$ bars, the last space always gets the value N. What you now have is a sequence of $n+N$ numbers with bars between consecutive distinct numbers and stars between consecutive equal numbers. For example, our example would gives $1|2*2|3|4*4|5$. Note that this gives a non-decreasing sequence of $n+N$ numbers. The corresponding non-decreasing sequence of n numbers for this arrangement of stars and bars is (2,4); that is we only take the numbers to the right of the stars. Because there are n stars, our sequence will have n numbers. It will be non-decreasing because it is a sub-sequence of the non-decreasing $n+N$ sequence. This shows that any arrangement of n stars and $N-1$ bars gives one nondecreasing sequence of n numbers.

Conversely, we can take any nondecreasing sequence of n numbers and combine it with the full set of numbers $\{1, 2, 3, \dots, N\}$ to form a set of $n+N$ numbers. Now rearrange these numbers in a nondecreasing order. Put a bar between consecutive distinct numbers in this set and a star between consecutive equal numbers in this set. Note that the number to the right of each star is an element of the original n-number sequence. This shows that any nondecreasing sequence of n numbers corresponds to an arrangement of n stars and $N-1$ bars.

Combining the two paragraphs above, we now know that the number of non-decreasing sequences is the same as the number of arrangement of n stars and $N-1$ bars, which is $\binom{n+N-1}{n}$.

Remark: There is also a method— which will not be discussed here, but can be inferred by finding the pattern of the sums that lead to the number of non-decreasing sequences as we increase the value of n— that would interestingly give the number of non-decreasing sequences as

$$\sum_{k_{n-1}=1}^{N} \cdots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} k_1.$$

This sum can be simplified into $\binom{n+N-1}{n}$ by the "parallel summation formula" which is well-known but we didn't discuss in class because this is not a class on combinatorics.