ECS 315: Probability and Random Processes **2015/1**

HW Solution 3 — Due: Sep 9, 9:19 AM (in tutorial session)

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Problem 1. (**Classical Probability and Combinatorics**) A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases.

(a) How many different designs are possible? [Montgomery and Runger, 2010, Q2-51]

(b) A specific design is randomly generated by the Web server when you visit the site. If you visit the site five times, what is the probability that you will not see the same design? [Montgomery and Runger, 2010, Q2-71]

Solution:

(a) By the multiplication rule, total number of possible designs

$$= 4 \times 3 \times 5 \times 3 \times 5 = \boxed{900}.$$

(b) From part (a), total number of possible designs is 900. The sample space is now the set of all possible designs that may be seen on five visits. It contains $(900)^5$ outcomes. (This is ordered sampling with replacement.)

The number of outcomes in which all five visits are different can be obtained by realizing that this is ordered sampling without replacement and hence there are $(900)_5$ outcomes. (Alternatively, On the first visit any one of 900 designs may be seen. On the second visit there are 899 remaining designs. On the third visit there are 898 remaining designs. On the fourth and fifth visits there are 897 and 896 remaining designs, respectively. From the multiplication rule, the number of outcomes where all designs are different is $900 \times 899 \times 898 \times 897 \times 896$.)

Therefore, the probability that a design is not seen again is

$$\frac{(900)_5}{900^5} \approx \boxed{0.9889.}$$

Problem 2. (**Classical Probability and Combinatorics**) A bin of 50 parts contains five that are defective. A sample of two parts is selected at random, without replacement. Determine the probability that both parts in the sample are defective. [Montgomery and Runger, 2010, Q2-49]

Solution: The number of ways to select two parts from 50 is $\binom{50}{2}$ and the number of ways to select two defective parts from the 5 defective ones is $\binom{5}{2}$ Therefore the probability is

$$\frac{\binom{5}{2}}{\binom{50}{2}} = \frac{2}{245} = \boxed{0.0082}.$$

Alternatively, if the two parts in the sample are selected one by one, then we may also consider their ordering as well. In such case, we use the formula for "ordered sampling without replacement" instead of "unordered sampling without replacement":

$$\frac{(5)_2}{(50)_2} = \frac{5 \times 4}{50 \times 49} = \frac{2}{245} = \boxed{0.0082}.$$

Problem 3. (**Classical Probability and Combinatorics**) We all know that the chance of a head (H) or tail (T) coming down after a fair coin is tossed are fifty-fifty. If a fair coin is tossed ten times, then intuition says that five heads are likely to turn up.

Calculate the probability of getting exactly five heads (and hence exactly five tails).

Solution: There are 2^{10} possible outcomes for ten coin tosses. (For each toss, there is two possibilities, H or T). Only $\binom{10}{5}$ among these outcomes have exactly heads and five tails. (Choose 5 positions from 10 position for H. Then, the rest of the positions are automatically T.) The probability of have exactly 5 H and 5 T is

$$\boxed{\frac{\binom{10}{5}}{2^{10}} \approx 0.246.}$$

Note that five heads and five tails will turn up more frequently than any other single combination (one head, nine tails for example) but the sum of all the other possibilities is much greater than the single 5 H, 5 T combination.

Problem 4. Let A and B be events for which $P(A)$, $P(B)$, and $P(A \cup B)$ are known. Express the following probabilities in terms of the three known probabilities above.

(a) $P(A \cap B)$

(b) $P(A \cap B^c)$

(c) $P(B \cup (A \cap B^c))$

(d) $P(A^c \cap B^c)$

Solution:

(a) $P(A \cap B) = \boxed{P(A) + P(B) - P(A \cup B)}$. This property is shown in class.

(b) We have seen[1] in class that $P(A \cap B^c) = P(A) - P(A \cap B)$. Plugging in the expression for $P(A \cap B)$ from the previous part, we have

$$P(A \cap B^c) = P(A) - (P(A) + P(B) - P(A \cup B)) = \boxed{P(A \cup B) - P(B)}.$$

Alternatively, we can start from scratch with the set identity $A \cup B = B \cup (A \cap B^c)$ whose union is a disjoint union. Hence,

$$P(A \cup B) = P(B) + P(A \cap B^c).$$

Moving $P(B)$ to the LHS finishes the proof.

(c) $P(B \cup (A \cap B^c)) = \boxed{P(A \cup B)}$ because $A \cup B = B \cup (A \cap B^c)$.

(d) $P(A^c \cap B^c) = \boxed{1 - P(A \cup B)}$ because $A^c \cap B^c = (A \cup B)^c$.

[1] This shows up when we try to derive the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. The key idea is that the set A can be expressed as a disjoint union between $A \cap B$ and $A \cap B^c$. Therefore, by finite additivity, $P(A) = P(A \cap B) + P(A \cap B^c)$. It is easier to visualize this via the Venn diagram.

Problem 5.

(a) Suppose that $P(A|B) = 0.4$ and $P(B) = 0.5$ Determine the following:

(i) $P(A \cap B)$

(ii) $P(A^c \cap B)$

[Montgomery and Runger, 2010, Q2-105]

(b) Suppose that $P(A|B) = 0.2$, $P(A|B^c) = 0.3$ and $P(B) = 0.8$ What is $P(A)$? [Montgomery and Runger, 2010, Q2-106]

Solution:

(a) Recall that $P(A \cap B) = P(A|B)P(B)$. Therefore,

(i) $P(A \cap B) = 0.4 \times 0.5 = \boxed{0.2.}$

(ii) $P(A^c \cap B) = P(B \setminus A) = P(B) - P(A \cap B) = 0.5 - 0.2 = \boxed{0.3.}$
Alternatively, $P(A^c \cap B) = P(A^c|B)P(B) = (1 - P(A|B))P(B) = (1 - 0.4) \times 0.5 = 0.3$.

(b) By the total probability formula, $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0.2 \times 0.8 + 0.3 \times (1 - 0.8) = \boxed{0.22.}$

Problem 6. Continue from Problem 5 in HW2.

Recall that, there, we consider a random experiment whose sample space is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Find the following probabilities.

(a) $P(A|B)$

(b) $P(B|A)$

(c) $P(B|A^c)$

Solution: In HW2, we have already found

$$\begin{aligned} P(A) &= P(\{a, b, c\}) = 0.1 + 0.1 + 0.2 = 0.4, \\ P(B) &= P(\{c, d, e\}) = 0.2 + 0.4 + 0.2 = 0.8, \text{ and} \\ P(A \cap B) &= P(\{c\}) = 0.2. \end{aligned}$$

Therefore, by definition,

(a) $P(A|B) \equiv \frac{P(A\cap B)}{P(B)} = \frac{0.2}{0.8} = \boxed{\frac{1}{4}}$ and

(b) $P(B|A) \equiv \frac{P(B\cap A)}{P(A)} = \frac{P(A\cap B)}{P(A)} = \frac{0.2}{0.4} = \boxed{\frac{1}{2}}$.

(c) DO <u>*NOT*</u> start with $P(B|A^c) = 1 - P(B|A)$. This is not one of the formulas for conditional probabilities. Here, we will have to go back to the definition:

$$P(B|A^c) = \frac{P(B\cap A^c)}{P(A^c)} = \frac{P(\{d,e\})}{P(\{d,e\})} = \boxed{1}.$$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 7.

(a) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Find the range of possible values for $P(A\cap B)$. Hint: Smaller than the interval $[0,1]$. [Capinski and Zastawniak, 2003, Q4.21]

(b) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Find the range of possible values for $P(A\cup B)$. Hint: Smaller than the interval $[0,1]$. [Capinski and Zastawniak, 2003, Q4.22]

Solution:

(a) We will try to derive general bounds for $P(A\cap B)$.

First, recall[2], from the lecture notes, that "$P(A\cap B)$ can not exceed $P(A)$ and $P(B)$":

$$P(A\cap B) \le \min\{P(A), P(B)\}. \tag{3.1}$$

On the other hand, we know that

$$P(A\cup B) = P(A) + P(B) - P(A\cap B). \tag{3.2}$$

Now, $P(A\cup B)$ is a probability and hence its value must be between 0 and 1:

$$0 \le P(A\cup B) \le 1 \tag{3.3}$$

[2] Again, to see this, note that $A\cap B \subset A$ and $A\cap B \subset B$. Hence, we know that $P(A\cap B) \le P(A)$ and $P(A\cap B) \le P(B)$.

Combining (3.3) with (3.2) gives

$$P(A) + P(B) - 1 \le P(A \cap B) \le P(A) + P(B). \tag{3.4}$$

The second inequality in (3.4) is not useful because (3.1) gives a better[3] bound. So, we will replace the second inequality with (3.1):

$$P(A) + P(B) - 1 \le P(A \cap B) \le \min\{P(A), P(B)\}. \tag{3.5}$$

Finally, $P(A \cap B)$ is also a probability and hence it must be between 0 and 1:

$$0 \le P(A \cap B) \le 1 \tag{3.6}$$

Combining (3.6) and (3.5), we have

$$\max\{(P(A) + P(B) - 1), 0\} \le P(A \cap B) \le \min\{P(A), P(B), 1\}.$$

Note that number one at the end of the expression above is not necessary because the two probabilities under minimization can not exceed 1 themselves. In conclusion,

$$\max\{(P(A) + P(B) - 1), 0\} \le P(A \cap B) \le \min\{P(A), P(B)\}.$$

Plugging in the value $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$ gives the range $\boxed{\left[\frac{1}{6}, \frac{1}{2}\right]}$.

Note that the upper-bound can be obtained by constructing an example which has $A \subset B$. The lower-bound can be obtained by considering an example where $A \cup B = \Omega$.

(b) We will try to derive general bounds for $P(A \cup B)$.

By monotonicity, because both A and B are subset of $A \cup B$, we must have

$$P(A \cup B) \ge \max\{P(A), P(B)\}.$$

On the other hand, we know, from the finite sub-additivity property, that

$$P(A \cup B) \le P(A) + P(B).$$

Therefore,

$$\max\{P(A), P(B)\} \le P(A \cup B) \le P(A) + P(B).$$

Being a probability, $P(A \cup B)$ must be between 0 and 1. Hence,

$$\max\{P(A), P(B), 0\} \le P(A \cup B) \le \min\{(P(A) + P(B)), 1\}.$$

[3] When we already know that a number is less than 3, learning that it is less than 5 does not give us any new information.

Note that number 0 is not needed in the aximization because the two probabilities involved are automatically ≥ 0 themselves.

In conclusion,

$$\max\{P(A), P(B)\} \leq P(A \cup B) \leq \min\{(P(A) + P(B)), 1\}.$$

Plugging in the value $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$, we have

$$P(A \cup B) \in \boxed{\left[\frac{1}{2}, \frac{5}{6}\right]}.$$

The upper-bound can be obtained by making $A \perp B$. The lower-bound is achieved when $B \subset A$.

Problem 8. Someone has rolled a fair dice twice. Suppose he tells you that "one of the rolls turned up a face value of six". What is the probability that the other roll turned up a six as well? [Tijms, 2007, Example 8.1, p. 244]

Hint: Note the followings:

(a) The answer is not $\frac{1}{6}$.

(b) Although there is no use of the word "give" or "conditioned on" in this question, the probability we seek is a conditional one. We have an extra piece of information because we know that the event "one of the rolls turned up a face value of six" has occurred.

(c) The question says "one of the rolls" without telling us which roll (the first or the second) it is referring to.

Solution: Let the sample space be the set $\{(i,j)|i,j = 1,\dots,6\}$, where i and j denote the outcomes of the first and second rolls, respectively. They are all equally likely; so each has probability of 1/36. The event of two sixes is given by $A = \{(6,6)\}$ and the event of at least one six is given by $B = (1,6),\dots,(5,6),(6,6),(6,5),\dots,(6,1)$. Applying the definition of conditional probability gives

$$P(A|B) = P(A \cap B)/P(B) = \frac{1/36}{11/36}.$$

Hence the desired probability is $\boxed{1/11}$.