ECS 315: Probability and Random Processes **2015/1**

HW Solution 13 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

x \ y	2	4	5
1	0.02	0.10	0.08
3	0.08	0.32	0.40

(a) Evaluate the following quantities:

(i) $\mathbb{E}[XY]$

(ii) $\mathbb{E}[(X-3)(Y-2)]$

(iii) $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)]$

(iv) $\operatorname{Cov}[X,Y]$

(v) $\rho_{X,Y}$

Hint: Write down the formulas then use `MATLAB` or `Excel` to compute them.

(b) Find $\rho_{X,X}$

(c) Calculate the following quantities using the values of $\operatorname{Var} X$, $\operatorname{Cov}[X,Y]$, and $\rho_{X,Y}$ that you got earlier.

(i) $\operatorname{Cov}[3X+4, 6Y-7]$

(ii) $\rho_{3X+4,6Y-7}$

(iii) $\operatorname{Cov}[X, 6X-7]$

(iv) $\rho_{X,6X-7}$

Solution: The `MATLAB` codes are provided in the file `P_XY_EVarCov.m`.

(a)

(i) From `MATLAB`, $\mathbb{E}[XY] = \boxed{11.16}$.

(ii) From `MATLAB`, $\mathbb{E}[(X-3)(Y-2)] = \boxed{-0.88}$.

(iii) From `MATLAB`, $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)] = \boxed{104}$.

(iv) From `MATLAB`, $\text{Cov}[X,Y] = \boxed{0.032}$.

(v) From `MATLAB`, $\rho_{X,Y} = \boxed{0.044677}$.

(b) $\rho_{X,X} = \frac{\text{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\text{Var}[X]}{\sigma_X^2} = \boxed{1}$.

(c)

(i) $\text{Cov}[3X+4, 6Y-7] = 3 \times 6 \times \text{Cov}[X,Y] \approx 3 \times 6 \times 0.032 \approx \boxed{0.576}$.

(ii) Note that

$$\begin{aligned}\rho_{aX+b,cY+d} &= \frac{\text{Cov}[aX+b, cY+d]}{\sigma_{aX+b}\sigma_{cY+d}} \\ &= \frac{ac\text{Cov}[X,Y]}{|a|\sigma_X |c|\sigma_Y} = \frac{ac}{|ac|}\rho_{X,Y} = \text{sign}(ac) \times \rho_{X,Y}.\end{aligned}$$

Hence, $\rho_{3X+4,6Y-7} = \text{sign}(3 \times 4)\rho_{X,Y} = \rho_{X,Y} = \boxed{0.0447}$.

(iii) $\text{Cov}[X, 6X-7] = 1 \times 6 \times \text{Cov}[X,X] = 6 \times \text{Var}[X] \approx \boxed{3.84}$.

(iv) $\rho_{X,6X-7} = \text{sign}(1 \times 6) \times \rho_{X,X} = \boxed{1}$.

Problem 2. Suppose $X \sim \text{binomial}(5, 1/3)$, $Y \sim \text{binomial}(7, 4/5)$, and $X \perp\!\!\!\perp Y$. Evaluate the following quantities.

(a) $\mathbb{E}[(X-3)(Y-2)]$

(b) $\text{Cov}[X,Y]$

(c) $\rho_{X,Y}$

Solution:

(a) First, because X and Y are independent, we have $\mathbb{E}[(X-3)(Y-2)] = \mathbb{E}[X-3]\,\mathbb{E}[Y-2]$. Recall that $\mathbb{E}[aX+b] = a\mathbb{E}[X]+b$. Therefore, $\mathbb{E}[X-3]\,\mathbb{E}[Y-2] = (\mathbb{E}[X]-3)(\mathbb{E}[Y]-2)$ Now, for Binomial(n,p), the expected value is np. So,

$$(\mathbb{E}[X]-3)(\mathbb{E}[Y]-2) = \left(5 \times \frac{1}{3} - 3\right)\left(7 \times \frac{4}{5} - 2\right) = -\frac{4}{3} \times \frac{18}{5} = \boxed{-\frac{24}{5}} = -4.8.$$

(b) $\text{Cov}[X,Y] = \boxed{0}$ because $X \perp\!\!\!\perp Y$.

(c) $\rho_{X,Y} = \boxed{0}$ because $\text{Cov}[X,Y] = 0$

Problem 3. Suppose we know that $\sigma_X = \frac{\sqrt{21}}{10}$, $\sigma_Y = \frac{4\sqrt{6}}{5}$, $\rho_{X,Y} = -\frac{1}{\sqrt{126}}$.

(a) Find $\text{Var}[X+Y]$.

(b) Find $\mathbb{E}\left[(Y-3X+5)^2\right]$. Assume $\mathbb{E}\left[Y-3X+5\right] = 1$.

Solution:

(a) First, we know that $\text{Var}\,X = \sigma_X^2 = \frac{21}{100}$, $\text{Var}\,Y = \sigma_Y^2 = \frac{96}{25}$, and $\text{Cov}[X,Y] = \rho_{X,Y} \times \sigma_X \times \sigma_Y = -\frac{2}{25}$. Now,

$$\begin{aligned}
\text{Var}[X+Y] &= \mathbb{E}\left[\left((X+Y) - \mathbb{E}[X+Y]\right)^2\right] = \mathbb{E}\left[\left((X-\mathbb{E}X) + (Y-\mathbb{E}Y)\right)^2\right] \\
&= \mathbb{E}\left[(X-\mathbb{E}X)^2\right] + 2\mathbb{E}\left[(X-\mathbb{E}X)(Y-\mathbb{E}Y)\right] + \mathbb{E}\left[(Y-\mathbb{E}Y)^2\right] \\
&= \text{Var}\,X + 2\text{Cov}[X,Y] + \text{Var}\,Y \\
&= \boxed{\frac{389}{100}} = 3.89.
\end{aligned}$$

Remark: It is useful to remember that

$$\text{Var}[X+Y] = \text{Var}\,X + 2\text{Cov}[X,Y] + \text{Var}\,Y.$$

Note that when X and Y are uncorrelated, $\text{Var}[X+Y] = \text{Var}\,X + \text{Var}\,Y$. This simpler formula also holds when X and Y are independence because independence is a stronger condition.

(b) First, we write

$$Y - aX - b = (Y - \mathbb{E}Y) - a(X - \mathbb{E}X) - \underbrace{(a\mathbb{E}X + b - \mathbb{E}Y)}_{c}.$$

Now, using the expansion

$$(u+v+t)^2 = u^2 + v^2 + t^2 + 2uv + 2ut + 2vt,$$

we have

$$\begin{aligned}
(Y-aX-b)^2 = (Y-\mathbb{E}Y)^2 + a^2(X-\mathbb{E}X)^2 + c^2 \\
- 2a(X-\mathbb{E}X)(Y-\mathbb{E}Y) - 2c(Y-\mathbb{E}Y) + 2a(X-\mathbb{E}X)c.
\end{aligned}$$

Recall that $\mathbb{E}[X - \mathbb{E}X] = \mathbb{E}[Y - \mathbb{E}Y] = 0$. Therefore,

$$\mathbb{E}\left[(Y - aX - b)^2\right] = \operatorname{Var} Y + a^2 \operatorname{Var} X + c^2 - 2a\operatorname{Cov}[X, Y]$$

Plugging back the value of c, we have

$$\boxed{\mathbb{E}\left[(Y - aX - b)^2\right] = \operatorname{Var} Y + a^2 \operatorname{Var} X + (\mathbb{E}[(Y - aX - b)])^2 - 2a\operatorname{Cov}[X, Y]}.$$

Here, $a = 3$ and $b = -5$. Plugging these values along with the given quantities into the formula gives

$$\mathbb{E}\left[(Y - aX - b)^2\right] = \boxed{\frac{721}{100}} = 7.21.$$

Problem 4. The input X and output Y of a system subject to random perturbations are described probabilistically by the joint pmf $p_{X,Y}(x, y)$, where $x = 1, 2, 3$ and $y = 1, 2, 3, 4, 5$. Let P denote the joint pmf matrix whose i,j entry is $p_{X,Y}(i, j)$, and suppose that

$$P = \frac{1}{71}\begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

(a) Find the marginal pmfs $p_X(x)$ and $p_Y(y)$.

(b) Find $\mathbb{E}X$

(c) Find $\mathbb{E}Y$

(d) Find $\operatorname{Var} X$

(e) Find $\operatorname{Var} Y$

Solution: All of the calculations in this question are simply plugging numbers into appropriate formula. The `MATLAB` codes are provided in the file `P_XY_marginal_2.m`.

(a) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 26/71, & x = 1 \\ 25/71, & x = 2 \\ 20/71, & x = 3 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.3662, & x = 1 \\ 0.3521, & x = 2 \\ 0.2817, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 13/71, & y=1 \\ 8/71, & y=2 \\ 21/71, & y=3 \\ 15/71, & y=4 \\ 14/71, & y=5 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.1831, & y=1 \\ 0.1127, & y=2 \\ 0.2958, & y=3 \\ 0.2113, & y=4 \\ 0.1972, & y=5 \\ 0, & \text{otherwise.} \end{cases}$$

(b) $\mathbb{E}X = \frac{136}{71} \approx 1.9155$

(c) $\mathbb{E}Y = \frac{222}{71} \approx 3.1268$

(d) $\operatorname{Var} X = \frac{3230}{5041} \approx 0.6407$

(e) $\operatorname{Var} Y = \frac{9220}{5041} \approx 1.8290$

Problem 5. A webpage server can handle r requests per day. Find the probability that the server gets more than r requests at least once in n days. Assume that the number of requests on day i is $X_i \sim \mathcal{P}(\alpha)$ and that $X_1, \dots, X_n$ are independent.

Solution: [Gubner, 2006, Ex 2.10]

$$\begin{aligned} P\left[\bigcup_{i=1}^{n} [X_i > r]\right] &= 1 - P\left[\bigcap_{i=1}^{n} [X_i \le r]\right] = 1 - \prod_{i=1}^{n} P[X_i \le r] \\ &= 1 - \prod_{i=1}^{n} \left(\sum_{k=0}^{r} \frac{\alpha^k e^{-\alpha}}{k!}\right) = \boxed{1 - \left(\sum_{k=0}^{r} \frac{\alpha^k e^{-\alpha}}{k!}\right)^n.} \end{aligned}$$

Extra Questions

Problem 6. Suppose $X \sim \text{binomial}(5, 1/3)$, $Y \sim \text{binomial}(7, 4/5)$, and $X \perp\!\!\!\perp Y$.

(a) A vector describing the pmf of X can be created by the MATLAB expression:

```
x = 0:5; pX = binopdf(x,5,1/3).
```

What is the expression that would give `pY`, a corresponding vector describing the pmf of Y?

(b) Use `pX` and `pY` from part (a), how can you create the joint pmf matrix in MATLAB? Do not use "for-loop", "while-loop", "if statement". Hint: Multiply them in an appropriate orientation.

(c) Use `MATLAB` to evaluate the following quantities. Again, do not use "for-loop", "while-loop", "if statement".

(i) $\mathbb{E}X$

(ii) $P[X = Y]$

(iii) $P[XY < 6]$

Solution: The `MATLAB` codes are provided in the file `P_XY_jointfromMarginal_indp.m`.

(a) `y = 0:7; pY = binopdf(y,7,4/5);`

(b) `P = pX.'*pY;`

(c)

(i) $\mathbb{E}X = \boxed{1.667}$

(ii) $P[X = Y] = \boxed{0.0121}$

(iii) $P[XY < 6] = \boxed{0.2727}$

Problem 7. Suppose $\text{Var}\, X = 5$. Find $\text{Cov}\,[X, X]$ and $\rho_{X,X}$.

Solution:

(a) $\text{Cov}\,[X, X] = \mathbb{E}\,[(X - \mathbb{E}X)(X - \mathbb{E}X)] = \mathbb{E}\,[(X - \mathbb{E}X)^2] = \text{Var}\, X = \boxed{5}$.

(b) $\rho_{X,X} = \frac{\text{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\text{Var}\, X}{\sigma_X^2} = \frac{\text{Var}\, X}{\text{Var}\, X} = \boxed{1}$.