ECS 315: Probability and Random Processes **2015/1**

HW Solution 12 — Due: Dec 2, 9:19 AM

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Problem 1. Let $X \sim \mathcal{E}(3)$.

(a) For each of the following function $g(x)$. Indicate whether the random variable $Y = g(X)$ is a continuous random variable.

(i) $g(x) = x^2$.

(ii) $g(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$.

(iii) $g(x) = \begin{cases} 4e^{-4x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

(iv) $g(x) = \begin{cases} x, & x \leq 5, \\ 5, & x > 5. \end{cases}$

(b) Repeat part (a), but now check whether the random variable $Y = g(X)$ is a discrete random variable.

Hint: As shown in class, to check whether Y is a continuous random variable, we may check whether $P[Y = y] = 0$ for any real number y. Because $Y = g(X)$, the probability under consideration is

$$P[Y = y] = P[g(X) = y] = P[X \in \{x : g(x) = y\}].$$

- At a particular y value, if there are at most countably many x values that satisfy $g(X) = y$, then
$$P[Y = y] = \sum_{x: g(x) = y} P[X = x].$$
When X is a continuous random variable, $P[X = x] = 0$ for any x. Therefore, $P[Y = y] = 0$.

  Conclusion: Suppose X is a continuous random variable and there are at most countably many x values that satisfy $g(X) = y$ for any y. Then, Y is a continuous random variable.

- Suppose at a particular y value, we have uncountably many x values that satisfy $g(X) = y$. Then, we can't rewrite $P[X \in \{x : g(x) = y\}]$ as $\sum_{x:g(x)=y} P[X = x]$ because axiom P3 of probability is only applicable to countably many disjoint sets (cases).

  When X is a continuous random variable, we can find probability by integrating its pdf:

$$P[Y = y] = P[X \in \{x : g(x) = y\}] = \int_{\{x:g(x)=y\}} f_X(x)dx.$$

  If we can find a y value whose integration above is > 0 , we can conclude right away that Y is not a continuous random variable.

Solution:

(a)

(i) $\boxed{\text{YES}}$.

When $y < 0$, there is no x that satisfies $g(x) = y$. When $y = 0$, there is exactly one x ($x = 0$) that satisfies $g(x) = y$. When $y > 0$, there is exactly two x ($x = \pm\sqrt{y}$) that satisfies $g(x) = y$.

Therefore, for any y, there are at most countably many x values that satisfy $g(X) = y$. Because X is a continuous random variable, we conclude that Y is also a continuous random variable.

(ii) $\boxed{\text{NO}}$. An easy way to see this is that there can be only two values out of the function $g(\cdot)$: 0 or 1. So, $Y = g(X)$ is a discrete random variable.

Alternatively, consider $y = 1$. We see that any $x \geq 0$, can make $g(x) = 1$. Therefore,

$$P[Y = 1] = P[X \geq 0].$$

For $X \sim \mathcal{E}(3)$, $P[X \geq 0] = 1 > 0$.

Because we found a y with $P[Y = y] > 0$. Y can not be a continuous random variable.

(iii) $\boxed{\text{YES}}$.

The plot of the function $g(x)$ may help you see the following facts: When $y > 4$ or $y < 0$, there is no x that gives $y = g(x)$. When $0 < y < 4$, there is exactly one x that satisfies $y = g(x)$. Because X is a continuous random variable, we can conclude that $P[Y = y]$ is 0 for $y \neq 0$.

When $y = 0$, any $x < 0$ would satisfy $g(x) = y$. So, $P[Y = 0] = P[X < 0]$. However, because $X \sim \mathcal{E}(3)$ is always positive. $P[X < 0] = 0$.

(iv) $\boxed{\text{NO}}$. Consider $y = 5$. We see that any $x \geq 5$, can make $g(x) = 5$. Therefore,

$$P[Y = 5] = P[X \geq 5].$$

For $X \sim \mathcal{E}(3)$,

$$P[X \geq 5] = \int_5^\infty 3e^{-3x} dx = e^{-15} > 0.$$

Because $P[Y = 5] > 0$, we conclude that Y can't be a continuous random variable.

(b) To check whether a random variable is discrete, we simply check whether it has a countable support. Also, if we have already checked that a random variable is continuous, then it can't also be discrete.

(i) $\boxed{\text{NO}}$. We checked before that it is a continuous random variable.

(ii) $\boxed{\text{YES}}$ as discussed in part (a).

(iii) $\boxed{\text{NO}}$. We checked before that it is a continuous random variable.

(iv) $\boxed{\text{NO}}$. Because X is positive, $Y = g(X)$ can be any positive number in the interval $(0, 5]$. The interval is uncountable. Therefore, Y is not discrete.

We have shown previously that Y is not a continuous random variable. Here. knowing that it is not discrete means that it is of the last type: mixed random variable.

Problem 2. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

$$\begin{array}{cc} x \backslash y & \begin{array}{ccc} 2 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{bmatrix} 0.02 & 0.10 & 0.08 \\ 0.08 & 0.32 & 0.40 \end{bmatrix} \end{array}$$

Evaluate the following quantities:

(a) The marginal pmf $p_X(x)$

(b) The marginal pmf $p_Y(y)$

(c) $\mathbb{E}X$

(d) $\operatorname{Var} X$

(e) $\mathbb{E}Y$

(f) $\operatorname{Var} Y$

(g) $P[XY < 6]$

(h) $P[X = Y]$

Solution: The `MATLAB` codes are provided in the file `P_XY_marginal.m`.

(a) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \boxed{\begin{cases} 0.2, & x = 1 \\ 0.8, & x = 3 \\ 0, & \text{otherwise.} \end{cases}}$$

(b) The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \boxed{\begin{cases} 0.1, & y = 2 \\ 0.42, & y = 4 \\ 0.48, & y = 5 \\ 0, & \text{otherwise.} \end{cases}}$$

(c) $\mathbb{E}X = \sum_x x p_X(x) = 1 \times 0.2 + 3 \times 0.8 = 0.2 + 2.4 = \boxed{2.6}$.

(d) $\mathbb{E}[X^2] = \sum_x x^2 p_X(x) = 1^2 \times 0.2 + 3^2 \times 0.8 = 0.2 + 7.2 = 7.4$. So, $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = 7.4 - (2.6)^2 = 7.4 - 6.76 = \boxed{0.64}$.

(e) $\mathbb{E}Y = \sum_y y p_Y(y) = 2 \times 0.1 + 4 \times 0.42 + 5 \times 0.48 = 0.2 + 1.68 + 2.4 = \boxed{4.28}$.

(f) $\mathbb{E}[Y^2] = \sum_y y^2 p_Y(y) = 2^2 \times 0.1 + 4^2 \times 0.42 + 5^2 \times 0.48 = 19.12$. So, $\operatorname{Var} Y = \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 = 19.12 - 4.28^2 = \boxed{0.8016}$.

(g) Among the 6 possible pairs of (x, y) shown in the joint pmf matrix, only the pairs $(1, 2)$, $(1, 4)$, $(1, 5)$ satisfy $xy < 6$. Therefore, $[XY < 6] = [X = 1]$ which implies $P[XY < 6] = P[X = 1] = \boxed{0.2}$.

(h) Among the 6 possible pairs of (x, y) shown in the joint pmf matrix, there is no pair which has $x = y$. Therefore, $P[X = Y] = \boxed{0}$.

Problem 3. The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the probability density function of X.

(b) What proportion of reactions is complete within 200 milliseconds?

Solution: See handwritten solution

Problem 4. Let a continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the probability density function of X is

$$f_X(x) = \begin{cases} 5, & 4.9 \leq x \leq 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability that a current measurement is less than 5 milliamperes.

(b) Find and plot the cumulative distribution function of the random variable X.

(c) Find the expected value of X.

(d) Find the variance and the standard deviation of X.

(e) Find the expected value of power when the resistance is 100 ohms?

Solution: See handwritten solution

Q3: pdf and cdf - chemical reaction

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$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $F_X(x)$ is a continuous function. Therefore, X is a continuous RV.

(a) $$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} -(-0.01) e^{-0.01x}, & x > 0, \\ 0, & x < 0. \end{cases} = \begin{cases} 0.01 e^{-0.01x}, & x > 0, \\ 0, & x < 0. \end{cases}$$

At $x = 0$, the derivative does not exist. Because this is just a point, we may assign $f_X(0)$ to be any arbitrary value. Here, we set $f_X(0) = 0$:

$$f_X(x) = \begin{cases} 0.01 e^{-0.01x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $$P[X < 200] = P[X \leq 200] = F_X(200) = 1 - e^{-0.01 \times 200} = 1 - e^{-2} \approx 0.8647.$$

Alternatively, $$P[X < 200] = \int_{-\infty}^{200} f_X(x)\,dx = \cancelto{0}{\int_{-\infty}^{0} f_X(x)\,dx} + \int_{0}^{200} f_X(x)\,dx$$

$$= \int_0^{200} 0.01 e^{-0.01x}\,dx = \left.\frac{0.01}{(-0.01)} e^{-0.01x}\right|_0^{200}$$

$$= \left(-e^{-0.01 \times 200}\right) - \left(-e^{-0.01 \times 0}\right) = -e^{-2} - (-1)$$

$$= 1 - e^{-2}$$

Q4: pdf, cdf, expected value, variance - current and power

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$$f_X(x) = \begin{cases} 5, & 4.9 \le x \le 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) $$P[X < 5] = \int_{-\infty}^{5} f_X(x)\,dx = \cancel{\int_{-\infty}^{4.9} \underbrace{f_X(x)}_{0}\,dx} + \int_{4.9}^{5} \underbrace{f_X(x)}_{5}\,dx$$

$$= 5x\Big|_{4.9}^{5} = 5(5-4.9) = 5\times 0.1 = 0.5$$

(b) $$F_X(x) = P[X \le x] = \int_{-\infty}^{x} f_X(t)\,dt$$

For $x < 4.9$, $f_X(t) = 0$ for all t inside $(0, -\infty)$.

Therefore, $F_X(x) = \int_{-\infty}^{x} 0\,dt = 0$.

For $4.9 \le x \le 5.1$, $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \cancel{\int_{-\infty}^{4.9} \underbrace{f_X(t)}_{0}\,dt} + \int_{4.9}^{x} \underbrace{f_X(t)}_{5}\,dt$$

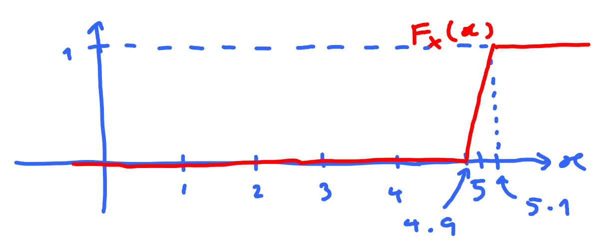
$$= 5t\Big|_{4.9}^{x} = 5\times(x-4.9) = 5x - 24.5.$$

For $x > 5.1$, $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \cancel{\int_{-\infty}^{4.9} \underbrace{f_X(t)}_{0}\,dt} + \int_{4.9}^{5.1} \underbrace{f_X(t)}_{5}\,dt + \cancel{\int_{5.1}^{x} \underbrace{f_X(t)}_{0}\,dt}$$

$$= 5t\Big|_{4.9}^{5.1} = 5\times(5.1-4.9) = 5\times 0.2 = 1.$$

Combining the three cases above, we have the complete description of the cdf:

$$F_X(x) = \begin{cases} 0, & x < 4.9, \\ 5x - 24.5, & 4.9 \le x \le 5.1, \\ 1, & x > 5.1 \end{cases}$$



Note that F_X is a continuous function. This is because it is the cdf of a continuous RV.

(c) $$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \cancel{\int_{0}^{4.9} x \underbrace{f_X(x)}_{0}\,dx} + \int_{4.9}^{5.1} x \underbrace{f_X(x)}_{5}\,dx + \cancel{\int_{5.1}^{\infty} x \underbrace{f_X(x)}_{0}\,dx}$$

5.1

$$= 5\,\frac{x^2}{2}\Bigg|_{4.9}^{5.1} = \frac{5}{2}\left(5.1^2 - 4.9^2\right) = \frac{5}{2}(5.1+4.9)(5.1-4.9) = \frac{5}{2}(10)(0.2)$$

$$= 5 \text{ mA}$$

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\mathbb{E}X = \frac{b+a}{2} = \frac{5.1+4.9}{2} = \frac{10}{2} = 5.$

(d) $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2$. From (c), we know that $\mathbb{E}X = 5$. So, to find $\operatorname{Var} X$, we need to find $\mathbb{E}[X^2]$.

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{4.9}^{5.1} x^2 \times 5\,dx = 5\,\frac{x^3}{3}\Bigg|_{4.9}^{5.1} = \frac{5}{3}\times\left(5.1^3 - 4.9^3\right)$$

$$= 25 + \frac{1}{300}.$$

Therefore, $\operatorname{Var} X = \left(25 + \frac{1}{300}\right) - 5^2 = \frac{1}{300} \approx 0.0033 \text{ (mA)}^2$

and $\sigma_X = \frac{1}{10\sqrt{3}}$ mA ≈ 0.0577 mA.

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\operatorname{Var} X = \frac{(b-a)^2}{12} = \frac{(5.1-4.9)^2}{12}$

$$= \frac{(0.2)^2}{12} = \frac{4}{100\times 12} = \frac{1}{300}.$$

(e) Recall that $P = IV = I \times I r = I^2 r$.

Here $I = X$. Therefore $P = X^2 r$ and

$$\mathbb{E}P = \mathbb{E}[X^2 r] = r\,\mathbb{E}[X^2] = 100 \times \left(25 + \frac{1}{300}\right) = 2500 + \frac{1}{3}$$

$$\approx 2.50033\times 10^3 \left[\underbrace{(\text{mA})^2\,\Omega}_{\text{m}^2\ \underbrace{\text{A}^2\Omega}_{\text{W}}}\right] \approx 2.5 \text{ mW}.$$

Caution: The current is in mA.

Extra Question

Problem 5. The input X and output Y of a system subject to random perturbations are described probabilistically by the joint pmf $p_{X,Y}(x,y)$, where $x = 1,2,3$ and $y = 1,2,3,4,5$. Let P denote the joint pmf matrix whose i,j entry is $p_{X,Y}(i,j)$, and suppose that

$$P = \frac{1}{71}\begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

(a) Find the marginal pmfs $p_X(x)$ and $p_Y(y)$.

(b) Find $\mathbb{E}X$

(c) Find $\mathbb{E}Y$

(d) Find $\operatorname{Var} X$

(e) Find $\operatorname{Var} Y$

Solution: All of the calculations in this question are simply plugging numbers into appropriate formula. The MATLAB codes are provided in the file `P_XY_marginal_2.m`.

(a) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 26/71, & x = 1 \\ 25/71, & x = 2 \\ 20/71, & x = 3 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.3662, & x = 1 \\ 0.3521, & x = 2 \\ 0.2817, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 13/71, & y = 1 \\ 8/71, & y = 2 \\ 21/71, & y = 3 \\ 15/71, & y = 4 \\ 14/71, & y = 5 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.1831, & y = 1 \\ 0.1127, & y = 2 \\ 0.2958, & y = 3 \\ 0.2113, & y = 4 \\ 0.1972, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

(b) $\mathbb{E}X = \frac{136}{71} \approx 1.9155$

(c) $\mathbb{E}Y = \frac{222}{71} \approx 3.1268$

(d) $\operatorname{Var} X = \frac{3230}{5041} \approx 0.6407$

(e) $\operatorname{Var} Y = \frac{9220}{5041} \approx 1.8290$