ECS 315: Probability and Random Processes	**2015/1**

HW 12 — Due: Dec 2, 9:19 AM

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Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)
The extra questions at the end are optional.

(c) Late submission will be rejected.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Let $X \sim \mathcal{E}(3)$.

(a) For each of the following function $g(x)$. Indicate whether the random variable $Y = g(X)$ is a continuous random variable.

(i) $g(x) = x^2$.

(ii) $g(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$.

(iii) $g(x) = \begin{cases} 4e^{-4x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

(iv) $g(x) = \begin{cases} x, & x \leq 5, \\ 5, & x > 5. \end{cases}$

(b) Repeat part (a), but now check whether the random variable $Y = g(X)$ is a discrete random variable.

Hint: As shown in class, to check whether Y is a continuous random variable, we may check whether $P[Y = y] = 0$ for any real number y. Because $Y = g(X)$, the probability under consideration is

$$P[Y = y] = P[g(X) = y] = P[X \in \{x : g(x) = y\}].$$

- At a particular y value, if there are at most countably many x values that satisfy $g(X) = y$, then
$$P[Y = y] = \sum_{x:g(x)=y} P[X = x].$$
When X is a continuous random variable, $P[X = x] = 0$ for any x. Therefore, $P[Y = y] = 0$.

  Conclusion: Suppose X is a continuous random variable and there are at most countably many x values that satisfy $g(X) = y$ for any y. Then, Y is a continuous random variable.

- Suppose at a particular y value, we have uncountably many x values that satisfy $g(X) = y$. Then, we can't rewrite $P[X \in \{x : g(x) = y\}]$ as $\sum_{x:g(x)=y} P[X = x]$ because axiom P3 of probability is only applicable to countably many disjoint sets (cases).

  When X is a continuous random variable, we can find probability by integrating its pdf:
$$P[Y = y] = P[X \in \{x : g(x) = y\}] = \int\limits_{\{x:g(x)=y\}} f_X(x)dx.$$
If we can find a y value whose integration above is > 0 , we can conclude right away that Y is not a continuous random variable.

Problem 2. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

x \ y	2	4	5
1	0.02	0.10	0.08
3	0.08	0.32	0.40

Evaluate the following quantities:

(a) The marginal pmf $p_X(x)$

(b) The marginal pmf $p_Y(y)$

(c) $\mathbb{E}X$

(d) $\operatorname{Var} X$

(e) $\mathbb{E}Y$

(f) $\operatorname{Var} Y$

(g) $P[XY < 6]$

(h) $P[X = Y]$

Problem 3. The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the probability density function of X.

(b) What proportion of reactions is complete within 200 milliseconds?

Problem 4. Let a continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the probability density function of X is

$$f_X(x) = \begin{cases} 5, & 4.9 \leq x \leq 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability that a current measurement is less than 5 milliamperes.

(b) Find and plot the cumulative distribution function of the random variable X.

(c) Find the expected value of X.

(d) Find the variance and the standard deviation of X.

(e) Find the expected value of power when the resistance is 100 ohms?

Extra Question

Here is an optional question for those who want more practice.

Problem 5. The input X and output Y of a system subject to random perturbations are described probabilistically by the joint pmf $p_{X,Y}(x, y)$, where $x = 1, 2, 3$ and $y = 1, 2, 3, 4, 5$. Let P denote the joint pmf matrix whose i,j entry is $p_{X,Y}(i, j)$, and suppose that

$$P = \frac{1}{71}\begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

(a) Find the marginal pmfs $p_X(x)$ and $p_Y(y)$.

(b) Find $\mathbb{E}X$

(c) Find $\mathbb{E}Y$

(d) Find $\operatorname{Var} X$

(e) Find $\operatorname{Var} Y$