ECS 315: Probability and Random Processes **2015/1**

HW Solution 11 — Due: Nov 25, 9:19 AM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

[Montgomery and Runger, 2010, Q4-97]

Solution: See handwritten solution

Problem 2. Consider each random variable X defined below. Let $Y = 1 + 2X$. (i) Find and sketch the pdf of Y and (ii) Does Y belong to any of the (popular) families discussed in class? If so, state the name of the family and find the corresponding parameters.

(a) $X \sim \mathcal{U}(0, 1)$

(b) $X \sim \mathcal{E}(1)$

(c) $X \sim \mathcal{N}(0, 1)$

Solution: See handwritten solution

Problem 3. Consider each random variable X defined below. Let $Y = 1 - 2X$. (i) Find and sketch the pdf of Y and (ii) Does Y belong to any of the (popular) families discussed in class? If so, state the name of the family and find the corresponding parameters.

(a) $X \sim \mathcal{U}(0, 1)$

(b) $X \sim \mathcal{E}(1)$

(c) $X \sim \mathcal{N}(0, 1)$

Solution: See handwritten solution

HW11 Q1: Exponential RV

Friday, November 21, 2014 9:02 AM

Let T be the time to failure (in hours)

We know that $T \sim \mathcal{E}(\lambda)$ where $\lambda = 3\times 10^{-4}$.

Therefore,

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Here, we want to find $P[T > 10^4]$.

We will first provide the general formula for $P[T > t]$.

For $T \sim \mathcal{E}(\lambda)$ and $t > 0$,

$$P[T > t] = \int_t^\infty f_T(\tau)\,d\tau = \int_t^\infty \lambda e^{-\lambda\tau}\,d\tau = -e^{-\lambda\tau}\Big|_t^\infty = e^{-\lambda t}$$

Therefore, $P[T > 10^4] = e^{-3\times 10^{-4}\times 10^4} = e^{-3} \approx 0.0498$

(b) $P[T \leq 7000] = 1 - P[T > 7000] = 1 - e^{-3\times 10^{-4}\times 7000} = 1 - e^{-2.1} \approx 0.8775$

Remark: In class, we have already shown that for $T \sim \mathcal{E}(\lambda)$,

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$P[T > t] = \begin{cases} e^{-\lambda t}, & t > 0, \\ 1, & \text{otherwise.} \end{cases}$$

These formula can be applied here directly as well.

HW11 Q2 Affine Transformation

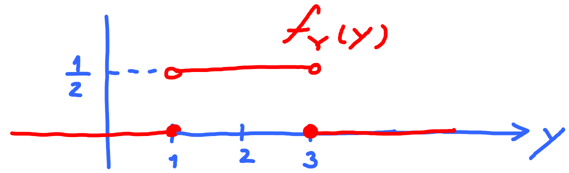
Tuesday, November 11, 2014 4:15 PM

We know that when $Y = aX+b$, we have $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$.

Here, $Y = \underbrace{2}_{a}X + \underbrace{1}_{b}$. Therefore, $f_Y(y) = \frac{1}{2} f_X\left(\frac{y-1}{2}\right)$.

(a) $X \sim \mathcal{U}(0,1) \Rightarrow f_X(x) = \begin{cases} 1, & 0<x<1, \\ 0, & \text{otherwise.} \end{cases}$

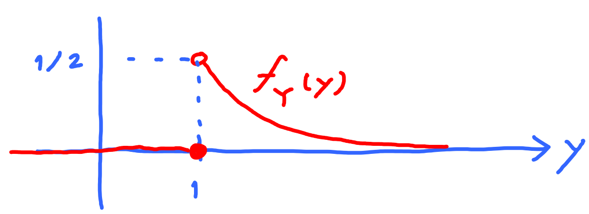
(i) Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} 1, & 0<\frac{y-1}{2}<1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1/2, & 1<y<3, \\ 0, & \text{otherwise.} \end{cases}$



(ii) Yes. $Y \sim \mathcal{U}(1,3)$

(b) $X \sim \mathcal{E}(1) \Rightarrow f_X(x) = \begin{cases} 1\times e^{-1\times x}, & x>0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} e^{-x}, & x>0, \\ 0, & \text{otherwise.} \end{cases}$

(i) Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} e^{-\left(\frac{y-1}{2}\right)}, & \frac{y-1}{2}>0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2}\sqrt{e}\, e^{-y/2}, & y>1, \\ 0, & \text{otherwise.} \end{cases}$



(ii) No. Although f_Y decays exponentially, the "exponential part" starts @ $y=1$ (not @ $y=0$). We may call this distribution a shifted exponential distribution. This distribution is quite useful for modeling output of a biological neuron with refractory period.

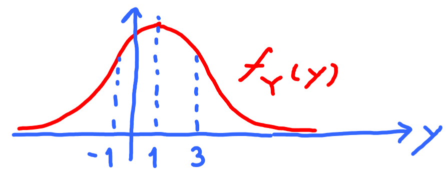
(c) We know that $X \sim \mathcal{N}(m, \sigma^2) \Rightarrow Y = \overset{2}{a}X + \overset{1}{b} \sim \mathcal{N}(am+b, a^2\sigma^2)$.

Plugging in $a=2$ and $b=1$, we have $Y \sim \mathcal{N}(2m+1, 4\sigma^2)$.

Plugging in $a=2$ and $b=1$, we have $Y \sim \mathcal{N}(2m+1,\ 4\sigma^2)$.

Here, $X \sim \mathcal{N}(\underbrace{0}_{m},\underbrace{1}_{\sigma^2})$. So, $Y \sim \mathcal{N}(2\times 0+1,\ 4\times 1) = \mathcal{N}(1, \underbrace{4}_{\sigma=2})$

(i) $f_Y(y) = \frac{1}{\sqrt{2\pi}\,2} e^{-\frac{1}{2}\left(\frac{y-1}{2}\right)^2}$



(ii) Yes. $Y \sim \mathcal{N}(1, 4)$

HW11 Q3 Affine Transformation

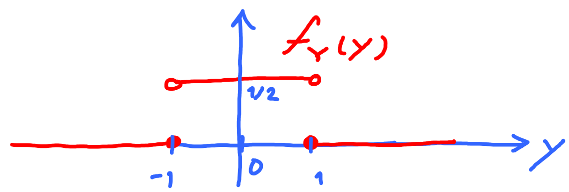
Tuesday, November 11, 2014 4:15 PM

We know that when $Y = aX + b$, we have $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$.

Here, $Y = \underbrace{-2}_{a}X + \underbrace{1}_{b}$. Therefore, $f_Y(y) = \frac{1}{2} f_X\left(\frac{1-y}{2}\right)$.

(a) $X \sim \mathcal{U}(0,1) \Rightarrow f_X(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$

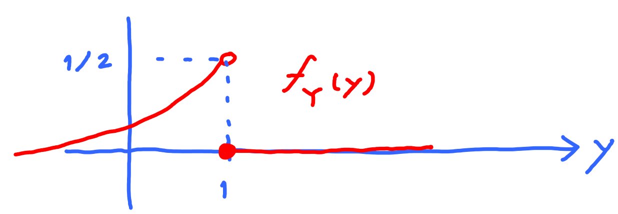
(i) Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} 1, & 0 < \frac{1-y}{2} < 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1/2, & -1 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$



(ii) Yes. $Y \sim \mathcal{U}(-1,1)$

(b) $X \sim \mathcal{E}(1) \Rightarrow f_X(x) = \begin{cases} 1 \times e^{-1 \times x}, & x > 0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$

(i) Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} e^{-\left(\frac{1-y}{2}\right)}, & \frac{1-y}{2} > 0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2\sqrt{e}} e^{y/2}, & y < 1, \\ 0, & \text{otherwise.} \end{cases}$



(ii) No. Although f_Y has exponential decay, its expression can not be rewritten in the form

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(c) We know that $X \sim \mathcal{N}(m, \sigma^2) \Rightarrow Y = \overset{-2}{a}X + \overset{1}{b} \sim \mathcal{N}(am+b, a^2\sigma^2)$.

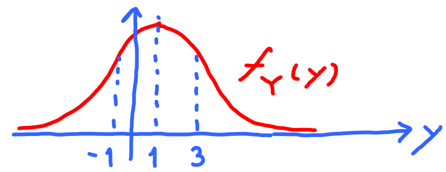
Plugging in $a = -2$ and $b = 1$, we have $Y \sim \mathcal{N}(1-2m, 4\sigma^2)$

Here $X \sim \mathcal{N}(0,1)$ so $Y \sim \mathcal{N}(1-2\times 0+1, 4\times 1) = \mathcal{N}(1, 4)$

Plugging in $a = -2$ and $b = 1$, we have $Y \sim \mathcal{N}(1-2m, 4\sigma^2)$

Here, $X \sim \mathcal{N}(\underset{m}{0}, \underset{\sigma^2}{1})$. So, $Y \sim \mathcal{N}(-2\times 0+1, 4\times 1) = \mathcal{N}(1, \underset{\sigma = 2}{4})$

(i) $f_Y(y) = \dfrac{1}{\sqrt{2\pi}\,2} e^{-\frac{1}{2}\left(\frac{y-1}{2}\right)^2}$



(ii) Yes. $Y \sim \mathcal{N}(1,4)$

Problem 4. Let $X \sim \mathcal{E}(5)$ and $Y = 2/X$.

(a) Check that Y is still a continuous random variable.

(b) Find $F_Y(y)$.

(c) Find $f_Y(y)$.

(d) (optional) Find $\mathbb{E}Y$.

Hint: Because $\frac{d}{dy}e^{-\frac{10}{y}} = \frac{10}{y^2}e^{-\frac{10}{y}} > 0$ for $y \neq 0$. We know that $e^{-\frac{10}{y}}$ is an increasing function on our range of integration. In particular, consider $y > 10/\ln(2)$. Then, $e^{-\frac{10}{y}} > \frac{1}{2}$. Hence,

$$\int_0^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}\frac{1}{2}dy = \int_{10/\ln 2}^\infty \frac{5}{y}dy.$$

Remark: To be technically correct, we should be a little more careful when writing $Y = \frac{2}{X}$ because it is undefined when $X = 0$. Of course, this happens with 0 probability; so it won't create any serious problem. However, to avoid the problem, we may define Y by

$$Y = \begin{cases} 2/X, & X \neq 0, \\ 0, & X = 0. \end{cases}$$

Solution: In this question, we have $Y = g(X)$ where the function g is defined by $g(x) = \frac{2}{x}$.

(a) First, we evaluate $P[Y = y] = P[g(X) = y]$.

- For each value of $y \neq 0$, there is only one x value that satisfies $y = g(x)$. (That x value is $x = \frac{2}{y}$.)
- When $y = 0$, there is no x value that satisfies $y = g(x)$.

In both cases, for each value of y, the number of solutions for $y = g(x)$ is countable. Therefore, we can write

$$P[Y = y] = \sum_{x:g(x)=y} P[X = x].$$

Here, $X \sim \mathcal{E}(5)$. Therefore, X is a continuous random variable and $P[X = x] = 0$ for any x. Hence,

$$P[Y = y] = \sum_{x:g(x)=y} 0 = 0.$$

Because $P[Y = y] = 0$ for all y, we conclude that Y is a continuous random variable.

(b) We consider two cases: "$y \le 0$" and "$y > 0$".

(i) Because $X > 0$, we know that $Y = \frac{2}{X}$ must be > 0 and hence, $F_Y(y) = 0$ for $y \le 0$. Note that Y can not be $= 0$. We need $X = \infty$ or $-\infty$ to make $Y = 0$. However, $\pm\infty$ are not real numbers therefore they are not possible X values.

(ii) For $y > 0$,

$$F_Y(y) = P[Y \le y] = P\left[\frac{2}{X} \le y\right] = P\left[X \ge \frac{2}{y}\right].$$

Note that, for the last equality, we can freely move X and y without worrying about "flipping the inequality" or "division by zero" because both X and y considered here are strictly positive. Now, for $X \sim \mathcal{E}(\lambda)$ and $x > 0$, we have

$$P[X \ge x] = \int_x^\infty \lambda e^{-\lambda t} dt = -e^{-\lambda t}\Big|_x^\infty = e^{-\lambda x}$$

Therefore,

$$F_Y(y) = e^{-5\left(\frac{2}{y}\right)} = e^{\frac{-10}{y}}.$$

Combining the two cases above we have

$$F_Y(y) = \boxed{\begin{cases} e^{-\frac{10}{y}}, & y > 0 \\ 0, & y \le 0 \end{cases}}$$

(c) Because we have already derived the cdf in the previous part, we can find the pdf via the cdf by $f_Y(y) = \frac{d}{dy}F_Y(y)$. This gives f_Y at all points except at $y = 0$ which we will set f_Y to be 0 there. (This arbitrary assignment works for continuous RV. This is why we need to check first that the random variable is actually continuous.) Hence,

$$f_Y(y) = \boxed{\begin{cases} \frac{10}{y^2} e^{-\frac{10}{y}}, & y > 0 \\ 0, & y \le 0. \end{cases}}$$

(d) We can find $\mathbb{E}Y$ from $f_Y(y)$ found in the previous part or we can even use $f_X(x)$

Method 1:

$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y) = \int_0^\infty y \frac{10}{y^2} e^{-\frac{10}{y}} dy = \int_0^\infty \frac{10}{y} e^{-\frac{10}{y}} dy$$

From the hint, we have

$$\mathbb{E}Y > \int_{10/\ln 2}^{\infty} \frac{10}{y} e^{-\frac{10}{y}} dy > \int_{10/\ln 2}^{\infty} \frac{10}{y}\frac{1}{2} dy = \int_{10/\ln 2}^{\infty} \frac{5}{y} dy$$
$$= 5 \ln y|_{10/\ln 2}^{\infty} = \infty.$$

Therefore, $\mathbb{E}Y = \boxed{\infty}$.

Method 2:

$$\mathbb{E}Y = \mathbb{E}\left[\frac{1}{X}\right] = \int_{-\infty}^{\infty} \frac{1}{x} f_X(x)\, dx = \int_{0}^{\infty} \frac{1}{x}\lambda e^{-\lambda x} dx > \int_{0}^{1} \frac{1}{x}\lambda e^{-\lambda x} dx$$
$$> \int_{0}^{1} \frac{1}{x}\lambda e^{-\lambda} dx = \lambda e^{-\lambda} \int_{0}^{1} \frac{1}{x} dx = \lambda e^{-\lambda} \ln x|_0^1 = \infty,$$

where the second inequality above comes from the fact that for $x \in (0,1)$, $e^{-\lambda x} > e^{-\lambda}$.

Problem 5. In wireless communications systems, fading is sometimes modeled by ***lognormal*** random variables. We say that a positive random variable Y is lognormal if $\ln Y$ is a normal random variable (say, with expected value m and variance σ^2).

(a) Check that Y is still a continuous random variable.

(b) Find the pdf of Y.

Hint: First, recall that the ln is the natural log function (log base e). Let $X = \ln Y$. Then, because Y is lognormal, we know that $X \sim \mathcal{N}(m, \sigma^2)$. Next, write Y as a function of X.

Solution:

Because $X = \ln(Y)$, we have $Y = e^X$. So, here, we consider $Y = g(X)$ where the function g is defined by $g(x) = e^x$.

(a) First, we evaluate $P[Y = y] = P[g(X) = y]$. Note that for each value of y, there is only one x value that satisfies $y = g(x)$. (That x value is $x = \ln(y)$.) So, the number of solutions for $y = g(x)$ is countable. Therefore, we can write

$$P[Y = y] = \sum_{x: g(x) = y} P[X = x].$$

Here, $X \sim \mathcal{N}(m, \sigma^2)$. Therefore, X is a continuous random variable and $P[X = x] = 0$ for any x. Hence,

$$P[Y = y] = \sum_{x:g(x)=y} 0 = 0.$$

Because $P[Y = y] = 0$ for all y, we conclude that Y is a continuous random variable.

(b) Start with $Y = e^X$. We know that exponential function gives strictly positive number. So, Y is always strictly positive. In particular, $F_Y(y) = 0$ for $y \le 0$.

Next, for $y > 0$, by definition, $F_Y(y) = P[Y \le y]$. Plugging in $Y = e^X$, we have

$$F_Y(y) = P\left[e^X \le y\right].$$

Because the exponential function is strictly increasing, the event $[e^X \le y]$ is the same as the event $[X \le \ln y]$. Therefore,

$$F_Y(y) = P[X \le \ln y] = F_X(\ln y).$$

Combining the two cases above, we have

$$F_Y(y) = \begin{cases} F_X(\ln y), & y > 0, \\ 0, & y \le 0. \end{cases}$$

Finally, we apply

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

For $y < 0$, we have $f_Y(y) = \frac{d}{dy} 0 = 0$. For $y > 0$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\ln y) = f_X(\ln y) \times \frac{d}{dy} \ln y = \frac{1}{y} f_X(\ln y). \tag{11.1}$$

Therefore,

$$f_Y(y) = \begin{cases} \frac{1}{y} f_X(\ln y), & y > 0, \\ 0, & y < 0. \end{cases}$$

At $y = 0$, because Y is a continuous random variable, we can assign any value, e.g. 0, to $f_Y(0)$. Then

$$f_Y(y) = \boxed{\begin{cases} \frac{1}{y} f_X(\ln y), & y > 0, \\ 0, & \text{otherwise.} \end{cases}}$$

Here, $X \sim \mathcal{N}(m, \sigma^2)$. Therefore,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

and

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{1}{2}\left(\frac{\ln(y)-m}{\sigma}\right)^2}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}}$$

Extra Questions

Problem 6. Cholesterol is a fatty substance that is an important part of the outer lining (membrane) of cells in the body of animals. Its normal range for an adult is 120–240 mg/dl. The Food and Nutrition Institute of the Philippines found that the total cholesterol level for Filipino adults has a mean of 159.2 mg/dl and 84.1% of adults have a cholesterol level below 200 mg/dl. Suppose that the cholesterol level in the population is normally distributed.

(a) Determine the standard deviation of this distribution.

(b) What is the value of the cholesterol level that exceeds 90% of the population?

(c) An adult is at moderate risk if cholesterol level is more than one but less than two standard deviations above the mean. What percentage of the population is at moderate risk according to this criterion?

(d) An adult is thought to be at high risk if his cholesterol level is more than two standard deviations above the mean. What percentage of the population is at high risk?

Solution: See handwritten solution

Problem 7. Consider a random variable X whose pdf is given by

$$f_X(x) = \begin{cases} cx^2, & x \in (1,2), \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = 4|X - 1.5|$.

(a) Find $\mathbb{E}Y$.

(b) Find $f_Y(y)$.

Solution: See handwritten solution

HW11 Q6: Gaussian RV - Cholesterol

Thursday, November 13, 2014 3:49 PM

Let X be the cholesterol level of a randomly chosen adult.

It is given that $X \sim \mathcal{N}(m, \sigma^2)$ where $m = 159.2$ mg/dl.

We also know that $P[X < 200] = 0.841$.

(a) $\Phi\left(\frac{200-m}{\sigma}\right) = 0.841 \Rightarrow \frac{200-m}{\sigma} \approx 1 \Rightarrow \sigma \approx 200 - m \approx 200 - 159.2 \approx 40.8$ mg/dl

From the Φ table,

$\Phi(0.99) \approx 0.8389$

$\Phi(1) \approx 0.8413$

0.841 is closer to 0.8413 than 0.8389

(b) We find x such that $\Phi\left(\frac{x-m}{\sigma}\right) = 0.9$.

From the Φ table, $\Phi(1.28) \approx 0.8997$

$\Phi(1.29) \approx 0.9015$

0.9 is closer to 0.8997

$\Rightarrow \frac{x-m}{\sigma} \approx 1.28 \Rightarrow x \approx 1.28\,\underset{\substack{\uparrow \\ 40.8 \\ \text{(from (a))}}}{\sigma} + \underset{\substack{\uparrow \\ 159.2}}{m} \approx 211.424$ mg/dl

(c) $P[m+\sigma < X < m+2\sigma] = F_X(m+2\sigma) - F_X(m+\sigma)$

$= \Phi\left(\frac{m+2\sigma-m}{\sigma}\right) - \Phi\left(\frac{m+\sigma-m}{\sigma}\right) = \Phi(2) - \Phi(1)$

$\approx 0.97725 - 0.8413 = 0.1359 \approx 13.59\%$

(d) $P[X > m+2\sigma] = 1 - F_X(m+2\sigma) = 1 - \Phi\left(\frac{m+2\sigma-m}{\sigma}\right) = 1 - \Phi(2)$

$\approx 1 - 0.97725 \approx 0.0228 = 2.28\%$

HW11 Q7: SISO

Saturday, November 22, 2014 11:54 PM

First, we need to find the constant c.

For any pdf, we know that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

Therefore, $\int_1^2 c x^2\,dx = c\int_1^2 x^2\,dx = c\left.\frac{x^3}{3}\right|_1^2 = c\left(\frac{8-1}{3}\right) = c\times\frac{7}{3}$ must $=1$.

Hence, $c = 3/7$.

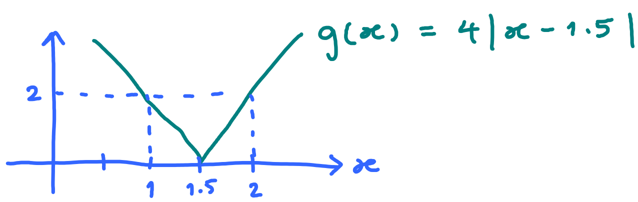
(a) $\mathbb{E}Y = \mathbb{E}\left[4|X-1.5|\right] = 4\int_1^2 |x-1.5|\,\frac{3}{7}x^2\,dx = \frac{12}{7}\int_1^2 |x-1.5|\,x^2\,dx$

$$|x-1.5| = \begin{cases} x-1.5, & x \geq 1.5 \\ 1.5-x, & x < 1.5 \end{cases}$$

$$= \frac{12}{7}\left(\int_1^{1.5}(1.5-x)\,x^2\,dx + \int_{1.5}^{2}(x-1.5)\,x^2\,dx\right) = \frac{57}{56}$$

(b) $Y = 4|X-1.5| = \begin{cases} 4X-6, & X \geq 1.5, \\ 6-4X, & X < 1.5 \end{cases} \equiv g(X)$

Let's plot the function $g(x)$:



First, let's check that Y is a cont. RV. This is easy to see from $g(x)$.
For each value of y, there are $\underbrace{\text{at most two}}_{\Downarrow}$ values of x that satisfy $y = g(x)$.

finite $\Rightarrow$ countable $\xRightarrow{X \text{ is a cont. RV}}$ $P[Y=y]=0\ \forall y$

$\Rightarrow Y$ is a cont. RV.

Step 1 : Find the cdf. Step 2 : $f_Y(y) = \frac{d}{dy}F_Y(y)$

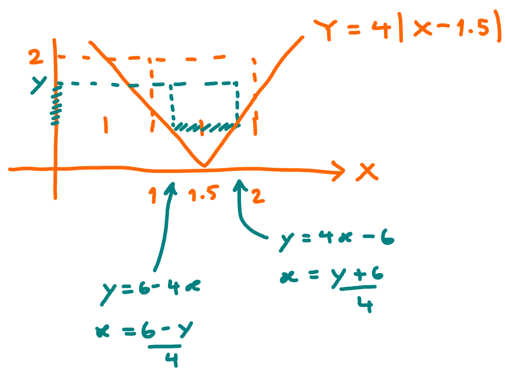
1.1 By construction (from 1.1), we know that $Y \geq 0$. Therefore,

$F_Y(y) = 0$ for $y < 0$.

2.1 This means $f_Y(y) = 0$ for $y < 0$. (*)

1.2 For $y = 0$, $F_Y(0) = P[Y \leq 0] = P[X = 0] \overset{\text{for cont. X.}}{=} 0$ (**)

1.3 For $y > 0$, Y ...



the event $[Y \le y]$ is the same as the event $\left[\frac{6-y}{4} \le X \le \frac{6+y}{4}\right]$.

Therefore,

for cont. X

$$F_Y(y) = P\left[\frac{6-y}{4} \le X \le \frac{6+y}{4}\right] = F_X\left(\frac{6+y}{4}\right) - F_X\left(\frac{6-y}{4}\right) \quad \text{when } y > 0.$$

(2.3) This implies

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{4}f_X\left(\frac{6+y}{4}\right) + \frac{1}{4}f_X\left(\frac{6-y}{4}\right) \quad \text{when } y > 0. \quad (***)$$

Plug-in $f_X(\cdot) = \frac{3}{7}(\cdot)^2$ when $1 < (\cdot) < 2$.

$1 < \frac{6+y}{4} < 2$ | $1 < \frac{6-y}{4} < 2$

$4 < 6+y < 8$ | $4 < 6-y < 8$

$-2 < y < 2$ | $-2 < -y < 2$

$-2 < y < 2$

Note again that this analysis is valid only for $y > 0$.

Therefore,

$$f_Y(y) = \begin{cases} \frac{1}{4} \times \frac{3}{7}\left(\left(\frac{6+y}{4}\right)^2 + \left(\frac{6-y}{4}\right)^2\right), & 0 < y < 2 \\ 0, & y \ge 2 \end{cases}$$

Combining (2.1) and (2.3), we have

$$f_Y(y) = \begin{cases} \frac{1}{4} \times \frac{3}{7}\left(\left(\frac{6+y}{4}\right)^2 + \left(\frac{6-y}{4}\right)^2\right), & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

At $y = 0$, we set $f_Y(0) = 0$. This is possible because Y is a continuous RV.

$$= \begin{cases} \frac{3}{224}\left(y^2 + 36\right), & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Check $\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^2 \frac{3}{224}\left(y^3 + 36y\right)dy = \frac{57}{56}$ ← same as part (a).