ECS 315: Probability and Random Processes **2010/1**

EXAM 2 — Name ID

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Date: Oct 13, 2010

Instructions:

(a) Including this cover page, there are 12 pages.

(b) ***One A4 sheet allowed***. Must be ***hand-written***. No small pieces of paper notes glued/attached on top of it. Indicate your name and ID on the upper right corner of the sheet.

(c) Read these instructions and the questions carefully.

(d) ***Closed book***. ***Closed notes***.

(e) ***Basic calculators***, e.g. FX-991MS, are ***permitted***, but borrowing is not allowed.

(f) Allocate your time wisely.

(g) The use of communication devices including mobile phones is prohibited in the examination room. Put all of your communication devices in your bag and leave it at the front of the examination room.

(h) Do not forget to write your first name and the last three digits of your ID in the spaces provide on the top of each examination page, starting from page 2.

(i) Unless specified otherwise, **write down all the steps** that you have done to obtain your answers. You may not get any credit even when your final answer is correct without showing how you get your answer.

(j) Some points are reserved for reducing answers into their simplest forms.

(k) When you are asked to "describe a random variable", say X, examples of your answer would be "X is $\mathcal{P}(\alpha)$ where $\alpha = 5$" or "X is a *continuous* random variable whose family we have not discussed in class."

(l) Do not cheat.

(m) Do not panic.

Problem 1. (31 pt) Consider a random variable X whose

$$f_X(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Use the above $f_X(x)$ for all parts in Problem 1.

(a) (1 pt) What command can you directly use to generate X in `MATLAB`?

(b) (1 pt*) Find $P\left[\sin(1000X + 12) = 0\right]$. (No partial credit.)

(c) (2 pt) Let

$$Y = \begin{cases} 1, & x > 0.2, \\ 0, & \text{otherwise.} \end{cases}$$

Describe the random variable Y.

(d) (2 pt + 1 pt*) Let

$$Z = \begin{cases} 0, & 0 < X < \frac{36}{49}, \\ 1, & \frac{36}{49} \le X < \frac{48}{49}, \\ 2, & \text{otherwise.} \end{cases}$$

Describe the random variable Z.

(e) (11 pt) Let

$$W = -\frac{1}{7}\ln X.$$

(i) (2 pt) Find the support S_W of W.

(ii) (3 pt) Find $\mathbb{E}W$
Hint: $\int \ln x dx = x\left((\ln x) - 1\right)$ and you may assume $0 \ln 0 = 0$.

(iii) (5 pt) Find the pdf of W. *Hint*: You should get a familiar pdf.

Remark: You should be able to directly solve part (ii) without going through part (iii). However, after you get the pdf of W from part (iii), you can find $\mathbb{E}W$ from the pdf and check whether you get the same answer as what you got in part (ii).

(iv) (1 pt) Describe the random variable W. (No partial credit.)

(f) (13 pt) Let $V = g(X)$ where

$$g(x) = \begin{cases} x, & x > \frac{1}{2}, \\ 0, & x \le \frac{1}{2}. \end{cases}$$

(i) (1 pt) Plot the function $y = g(x)$.

(ii) (5 pt) Find $\operatorname{Var} V$.

(iii) (4 pt) Plot the pdf of V.

(iv) (3 pt) Plot the cdf of V.

Problem 2. (15 pt) Suppose a random variable X has density

$$f_X(x) = ce^{-|x|} + \frac{1}{2}\delta(x) + \frac{1}{3}\delta(x-5).$$

(a) (3 pt) Find c.

(b) (2 pt) Find $P[X = 1]$

(c) (3 pt) Find $P[X \leq 0]$

(d) (5 pt) Find $\mathbb{E}\left[X^2\right]$

(e) (2 pt*) Find the pdf of $Y = X^2$.

Problem 3. (22 pt) Random variables X and Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} cxy^2, & 0 \le x \le 2, 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (3 pt) Find the constant c.

(b) (4 pt) Find $P[X > Y]$.

(c) (5 pt) Find $\operatorname{Var} X$.

(d) (2 pt) Are X and Y independent?

(e) (2 pt) Find $\text{Cov}[X, Y]$.

(f) (2 pt) Find $\mathbb{E}[Y|X]$.

(g) (2 pt) Find $\text{Var}[Y|X]$.

(h) (2 pt) Find $\text{Var}[X + Y]$.

Problem 4. (27 pt) Let $X \sim \mathcal{P}(5)$, and suppose that when $X = k$, Y will be Binomial$(k, 1/4)$. Note that when $X = 0$, Y becomes deterministic and always takes the value 0.

(a) (2 pt) Find $p_{Y|X}(0|5)$.

(b) (2 pt) Find $p_{X,Y}(5, 0)$.

(c) (2 pt + 1 pt*) Find $p_Y(0)$.

(d) (2 pt) Find $p_{X|Y}(5|0)$.

(e) (3 pt) Find $\mathbb{E}[Y|X = k]$ for $k = 0, 1, 2, 3, \ldots$.

(f) (3 pt) Find $\mathbb{E}[Y]$.

(g) (5 pt) Find $\mathbb{E}\left[Y^2\right]$.

(h) (1 pt) Find $\operatorname{Var}[Y]$.

(i) (2 pt) Find $\mathbb{E}\left[\operatorname{Var}[Y|X]\right]$.

(j) (1 pt) Find $\operatorname{Var}\left[\mathbb{E}\left[Y|X\right]\right]$. (This is the same as $\operatorname{Var} Z$ where $Z = \mathbb{E}\left[Y|X\right]$.)

(k) (2 pt) Find $\mathbb{E}\left[XY\right]$.

(l) (1 pt*) Find $\mathbb{E}\left[\frac{XY}{X+1}\right]$. (No partial credit. Use the space provided on p. 12 for calculation.)

For the questions on this page, there will be no partial credit. You only need to give the final answer. Use the space provided on the next page for you calculation. ***Put only your final answers on this page***.

Problem 5. (3 pt) Two trains, A and B arrive *independently* at a train station at times T_1 and T_2, respectively, where the arrival times are equally likely to occur anywhere within the time interval from 1PM to 3PM. Once at the train station, each one of the trains will stay for 15 min before departing.

(a) (1 pt) Calculate the probability that the two trains will meet.

(b) (1 pt) Calculate the probability that the two trains will meet ***and*** that A arrives before B.

(c) (1 pt) Calculate the probability that the two trains will meet ***given*** that A arrives before B.

Problem 6. (1 pt*) Let $X \sim \text{Binomial}(1800, 1/3)$. We can closely approximate the probability $P[590 < X \leq 610]$ by $2\Phi(z)-1$ where Φ is the standard normal cdf. Find z.

Problem 7. (1 pt) A message, sent from a transmitter to a receiver, is received error-free with probability 0.2. If a message is received with an error, it is retransmitted (again and again until it is correctly received). Let the random variable N be the number of transmissions of the message until it is received error-free. Let $W = 3N + \sqrt{3}$. Find $\text{Cov}[N, W]$.

Problem 8. (1 pt) Do not forget to submit your formula sheet with your final exam.

This page is intentionally left blank. Use it for your calculation.