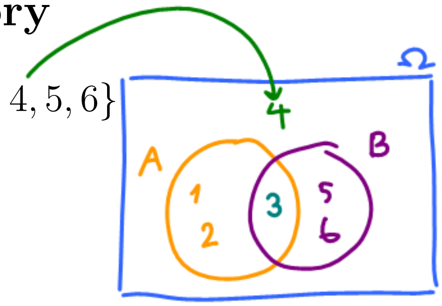
2 Review of Set Theory

Example 2.1. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$

Let $A = \{1, 2, 3\}$

$B = \{3, 5, 6\}$



2.2. Venn diagram is very useful in set theory. It is often used to portray relationships between sets. Many identities can be read out simply by examining Venn diagrams.

2.3. If ω is a member of a set A, we write $\omega \in A$.

Definition 2.4. Basic set operations (set algebra)

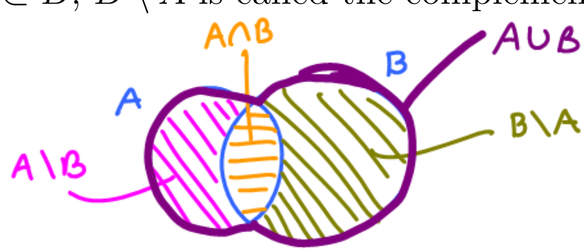
- Complementation: $A^c = \{\omega : \omega \notin A\}$.

$A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = \bigcup_{k=1}^{n} A_k$

- Union: $A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$
  - Here "or" is inclusive; i.e., if $\omega \in A$, we permit ω to belong either to A or to B or to both.
- Intersection: $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

$\bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \cdots \cap A_n$

  - Hence, $\omega \in A$ if and only if ω belongs to both A and B.
  - $A \cap B$ is sometimes written simply as AB.
- The ***set difference*** operation is defined by $B \setminus A = B \cap A^c$.
  - $B \setminus A$ is the set of $\omega \in B$ that do not belong to A.
  - When $A \subset B$, $B \setminus A$ is called the complement of A in B.



reading assignment

2.5. Basic Set Identities:

Compare these with Boolean algebra identities:

OR $\leftrightarrow \cup$

AND $\leftrightarrow \cap$

NOT $\leftrightarrow (\ )^c$

- Idempotence: $(\mathrm{A}^c)^c = \mathrm{A}$
- Commutativity (symmetry):

$$A \cup B = B \cup A\ ,\ A \cap B = B \cap A$$

- Associativity:
  - $A \cap (B \cap C) = (A \cap B) \cap C$
  - $A \cup (B \cup C) = (A \cup B) \cup C$
- Distributivity
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **de Morgan laws**
  - $(A \cup B)^c = A^c \cap B^c$
  - $(A \cap B)^c = A^c \cup B^c$

2.6. Disjoint Sets:

$A \perp\!\!\!\perp B \longrightarrow$ A and B are independent events

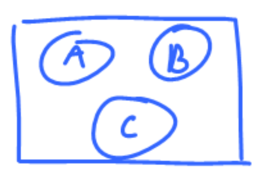
- Sets A and B are said to be ***disjoint*** ($A \perp B$) if and only if $A \cap B = \emptyset$. (They do not share member(s).)
- A collection of sets $(A_i : i \in I)$ is said to be (mutually) ***pairwise*** ***disjoint*** or mutually exclusive [9, p. 9] if and only if $\mathrm{A}_i \cap \mathrm{A}_j = \emptyset$ when $i \neq j$.

Example 2.7. Sets A, B, and C are pairwise disjoint if

$A \cap B = \emptyset$

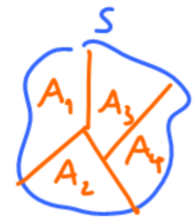
$B \cap C = \emptyset$

$A \cap C = \emptyset$



2.8. For a set of sets, to avoid the repeated use of the word "set", we will call it a **collection/class/family** of sets.

Definition 2.9. Given a set S, a collection $\Pi = (A_\alpha : \alpha \in I)$ of subsets[2] of S is said to be a ***partition*** of S if



"exhaustive"

(a) $S = \bigcup A_{\alpha \in I}$ and The union of all the sets in the collection is S itself.

(b) For all $i \neq j$, $A_i \perp A_j$ (pairwise disjoint). The sets in the collection are disjoint.

"mutually exclusive"

Remarks:

- The subsets A_α, $\alpha \in I$ are called the **parts** of the partition.
- A part of a partition may be empty, but usually there is no advantage in considering partitions with one or more empty parts.

Example 2.10 (Slide:maps).

Example 2.11. Let E be the set of students taking ECS315

A = set of students who will get A

B+ = " " B+

B =

C+ =

C =

D+ =

$\emptyset$ = D =

$\emptyset$ = F =

$\emptyset$ = W = " " withdraw

Definition 2.12. The ***cardinality*** (or size) of a collection or set A, denoted $|A|$, is the number of elements of the collection. This number may be finite or infinite.

$|A| \in \mathbb{N} \cup \{0\}$

- A *finite* set is a set that has a finite number of elements.
- A set that is not finite is called *infinite*.

- Countable sets:

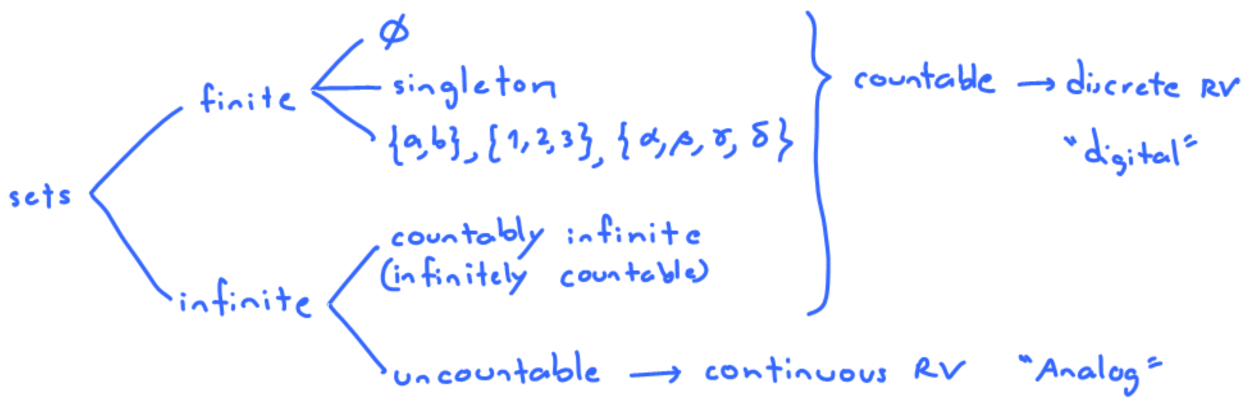
[2] In this case, the subsets are indexed or labeled by α taking values in an index or label set I

- ◦ Empty set and finite sets are automatically countable.
- ◦ An infinite set A is said to be ***countable*** if the elements of A can be enumerated or listed in a sequence: $a_1, a_2, \dots$.

countably infinite = countable + infinite

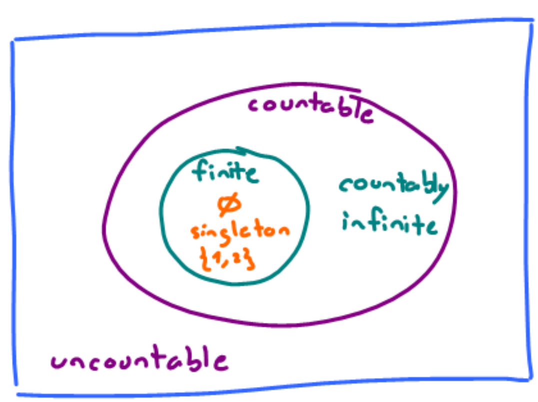
- A ***singleton*** is a set with exactly one element.
  - ◦ Ex. $\{1.5\}$, $\{.8\}$, $\{\pi\}$, {apple}, {n}, $\{\alpha\}$
  - ◦ *Caution*: Be sure you understand the difference between the outcome -8 and the event $\{-8\}$, which is the set consisting of the single outcome -8.

2.13. We can categorize sets according to their cardinality:



Example 2.14. Examples of countably infinite sets:

- the set $\mathbb{N} = \{1, 2, 3, \dots\}$ of natural numbers,
- the set $\{2k : k \in \mathbb{N}\}$ of all even numbers, $\{2, 4, 6, 8, 10, \dots\}$
- the set $\{2k - 1 : k \in \mathbb{N}\}$ of all odd numbers, $\{1, 3, 5, 7, 9, \dots\}$
- the set $\mathbb{Z}$ of integers, $\{\overset{a_1}{0}, \overset{a_2}{-1}, 1, -2, 2, -3, 3, -4, 4, \dots\}$



Set Theory	Probability Theory
Set	Event
Universal set	Sample Space (Ω)
Element	Outcome (ω)

Table 1: The terminology of set theory and probability theory

	Event Language
A	A occurs
A^c	A does not occur
$A \cup B$	Either A or B occur
$A \cap B$	Both A and B occur

Table 2: Event Language

Example 2.15. Example of uncountable sets[3]:

- $\mathbb{R} = (-\infty, \infty)$
- interval $[0, 1]$
- interval $(0, 1]$
- $(2, 3) \cup [5, 7)$

$(2,3) \cap (2.5, 5)$

$\cancel{(2,3] \cap [3,5) = \{3\}}$

Definition 2.16. Probability theory renames some of the terminology in set theory. See Table 1 and Table 2.

- Sometimes, ω's are called states, and Ω is called the state space.

2.17. Because of the mathematics required to determine probabilities, probabilistic methods are divided into two distinct types, discrete and continuous. A discrete approach is used when the number of experimental outcomes is finite (or infinite but countable). A continuous approach is used when the outcomes are continuous (and therefore infinite). It will be important to keep in mind which case is under consideration since otherwise, certain paradoxes may result.

[3] We use a technique called diagonal argument to prove that a set is not countable and hence uncountable.