Probability and Random Processes

ECS 315

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10 Continuous Random Variables

Office Hours:

BKD, 4th floor of Sirindhralai building

Monday	9:30-10:30
Monday	14:00-16:00
Thursday	16:00-17:00

`rand` function

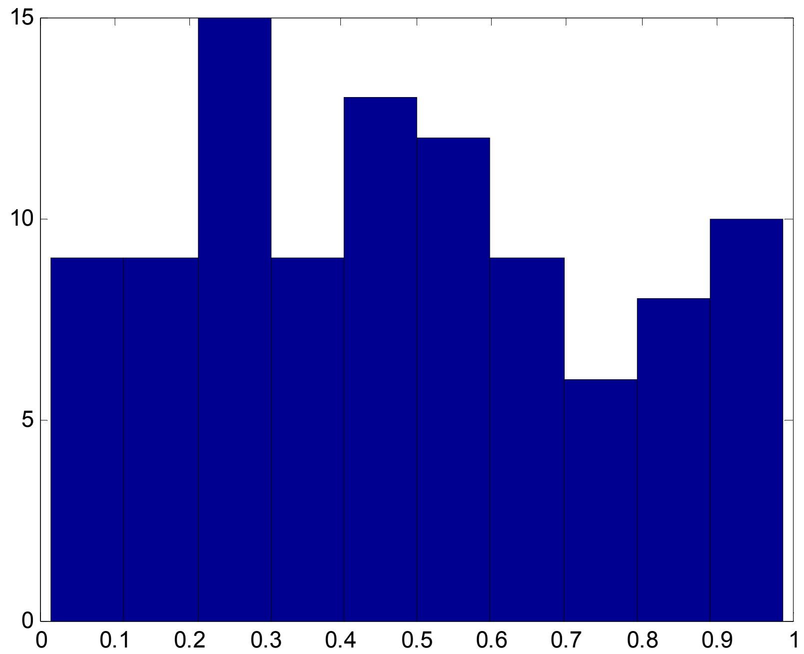
- Generate an array of uniformly distributed pseudorandom numbers.
  - The pseudorandom values are drawn from the **standard uniform distribution** on the open **interval (0,1)**.
- `rand` returns a scalar.
- `rand(m,n)` or `rand([m,n])` returns an *m*-by-*n* matrix.
  - `rand(n)` returns an *n*-by-*n* matrix

```
>> rand

ans =

    0.3816

>> rand(10,2)

ans =

    0.7655    0.6551
    0.7952    0.1626
    0.1869    0.1190
    0.4898    0.4984
    0.4456    0.9597
    0.6463    0.3404
    0.7094    0.5853
    0.7547    0.2238
    0.2760    0.7513
    0.6797    0.2551
```

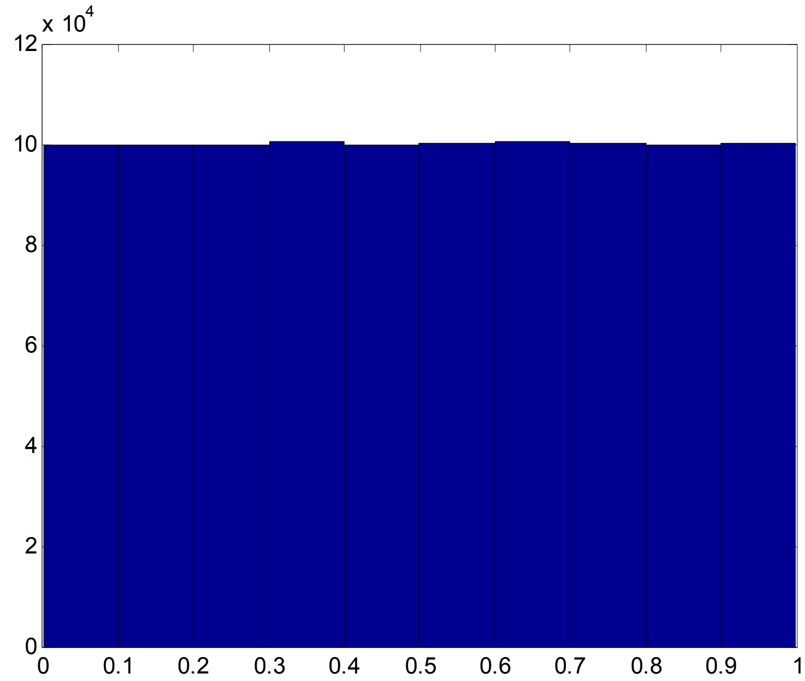
`rand` function: Histogram

- The generation is **unbiased** in the sense that "any number in the range is **as likely to occur** as another number."
- Histogram is flat over the interval (0,1).

`hist(rand(1,100))`

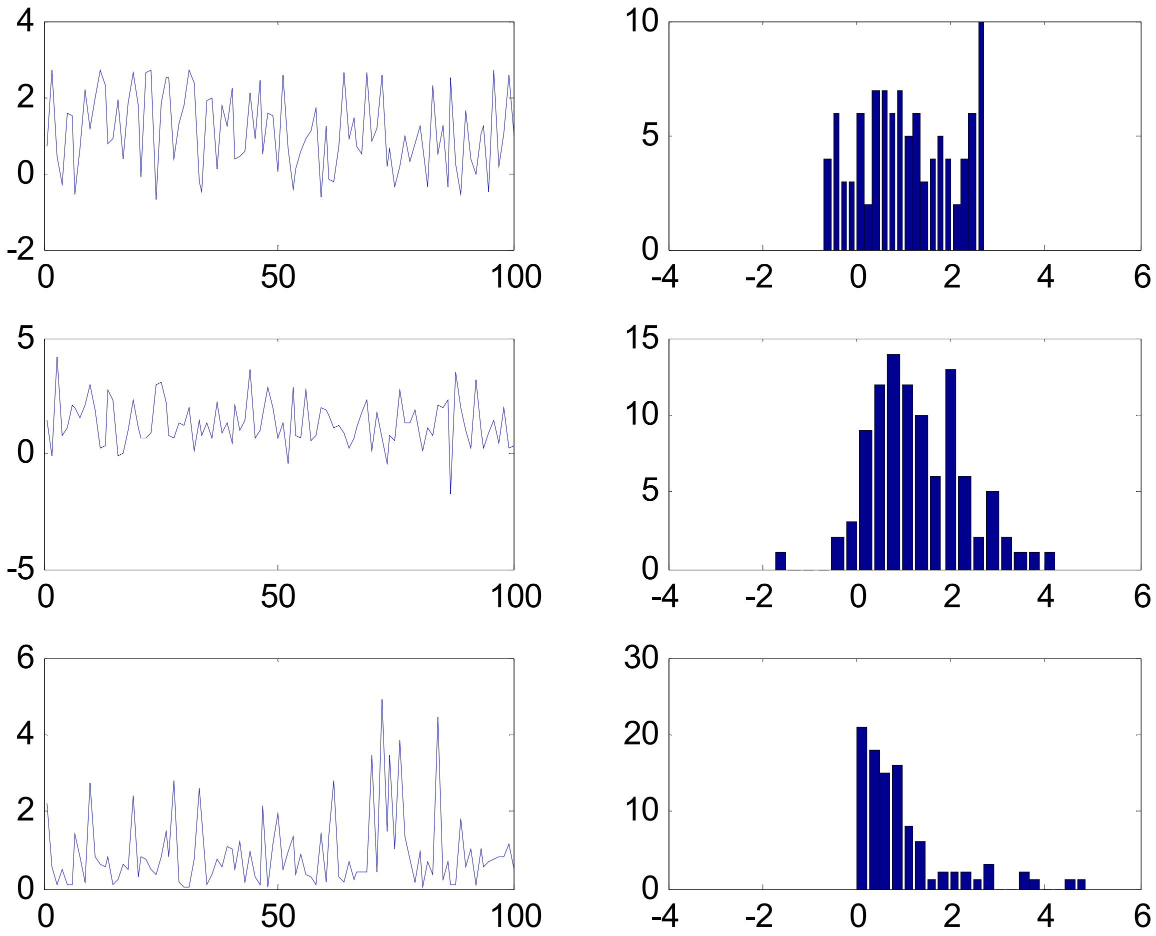


`hist(rand(1,1e6))`



Roughly the same height

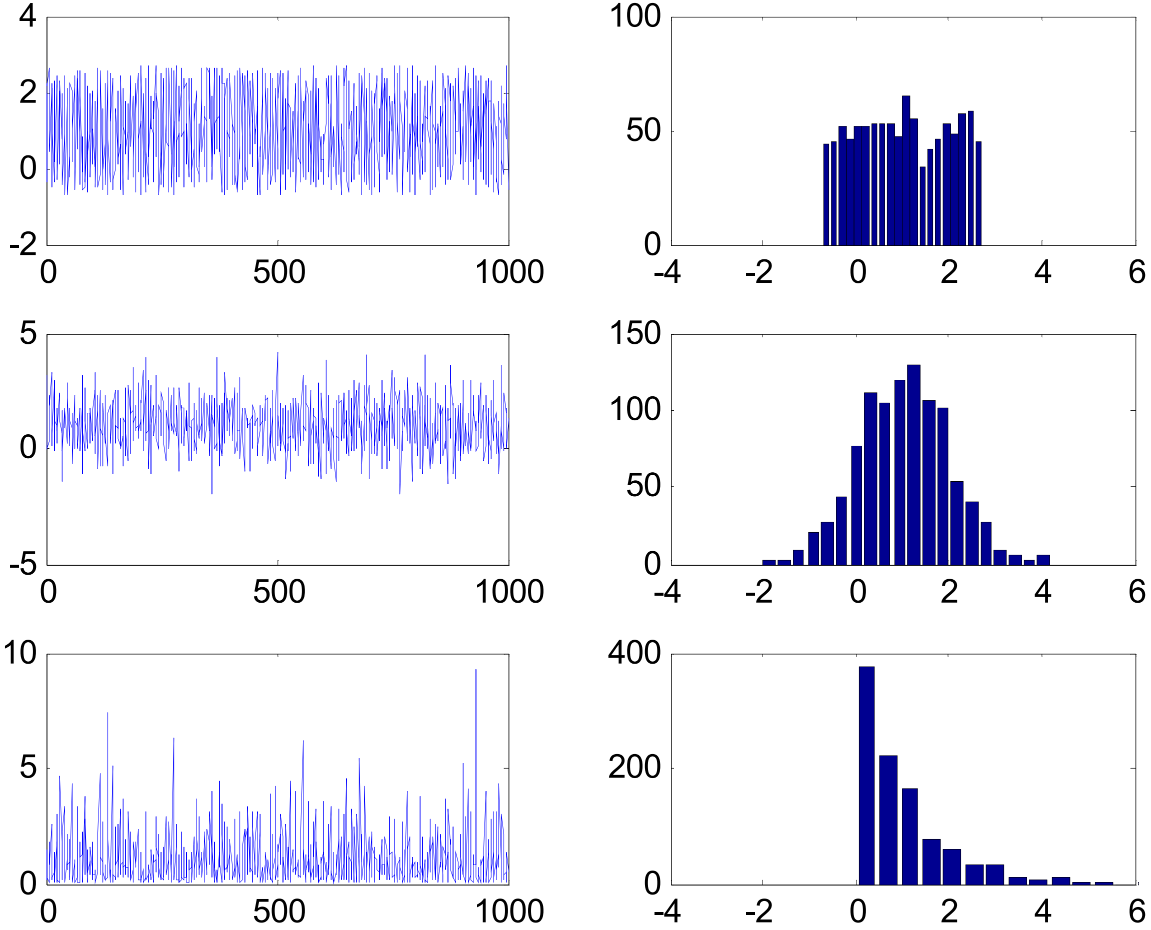
Three Important Continuous RVs



Mean = 1
Std = 1
N = 100

[IntroThreeContinuousRV.m]

Three Important Continuous RVs

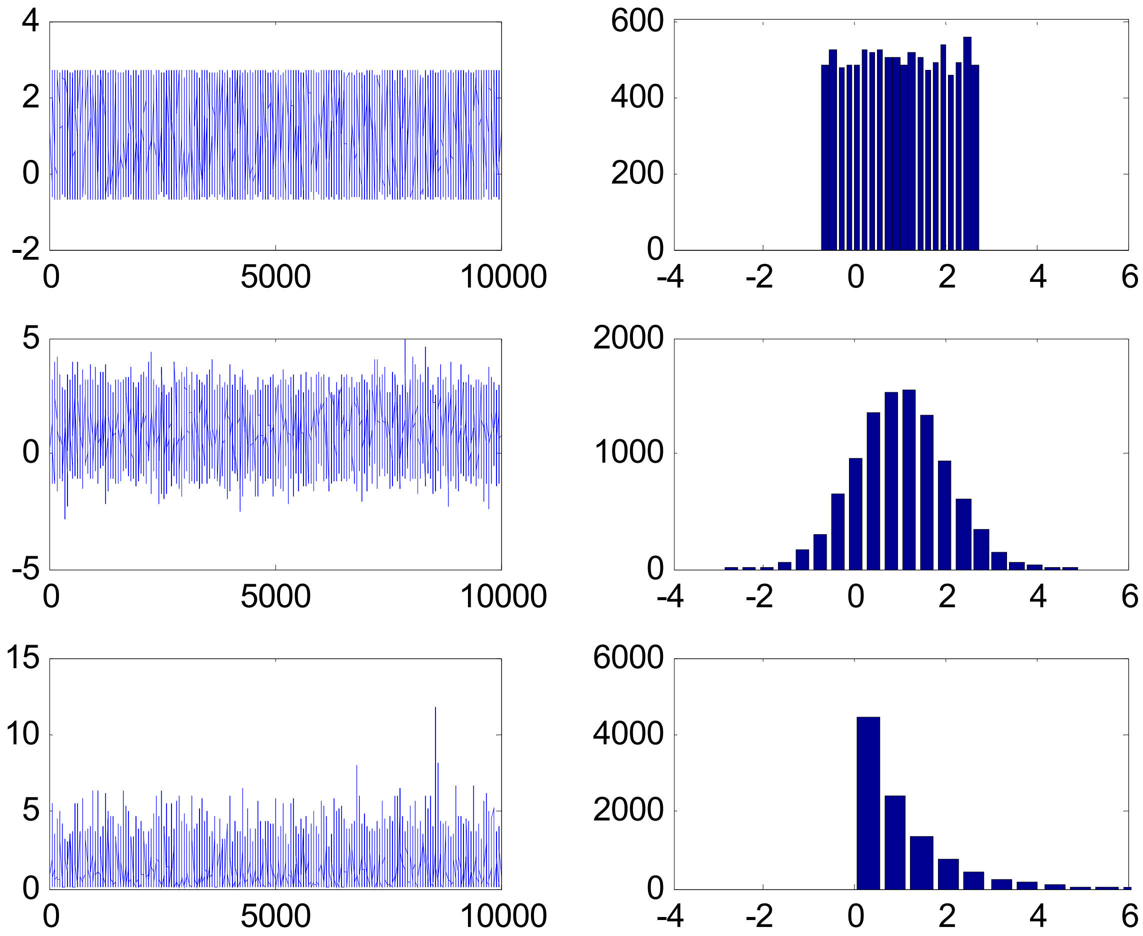


Mean = 1

Std = 1

N = 1,000

Three Important Continuous RVs

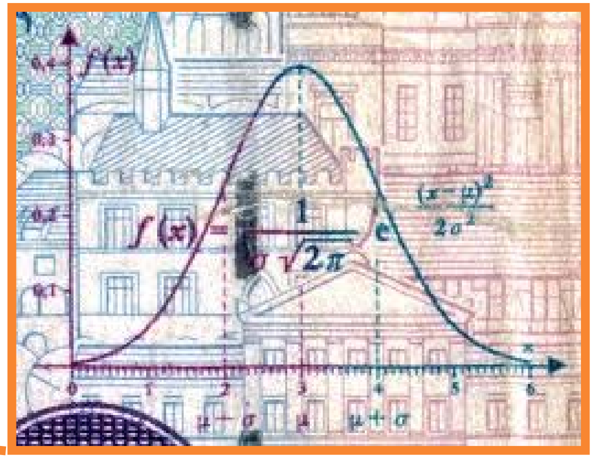


Mean = 1

Std = 1

N = 10,000

Johann Carl Friedrich Gauss

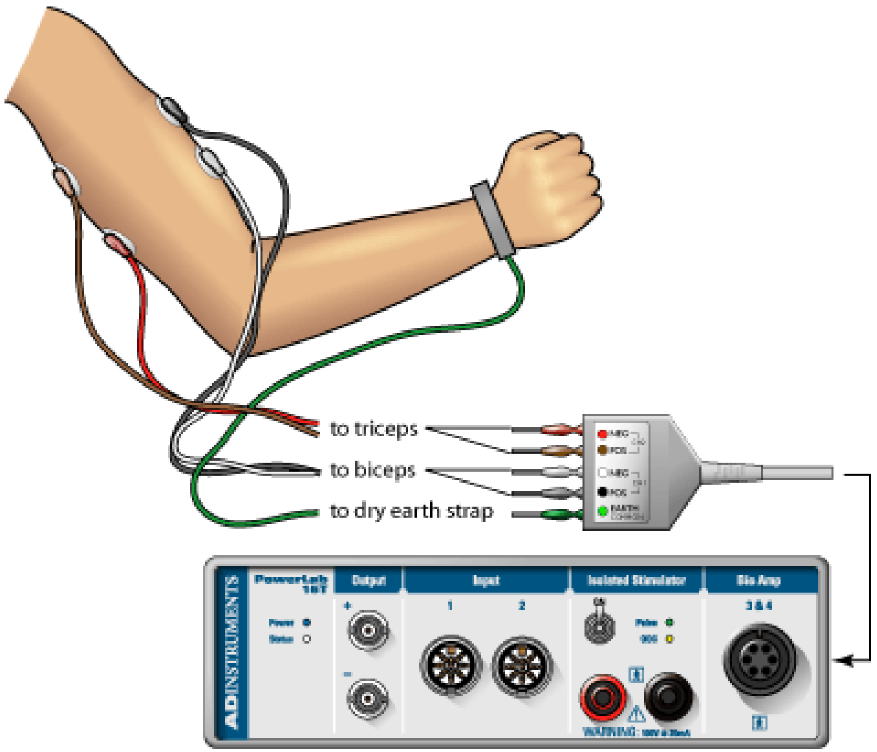


German 10-Deutsche Mark Banknote (1993; discontinued)

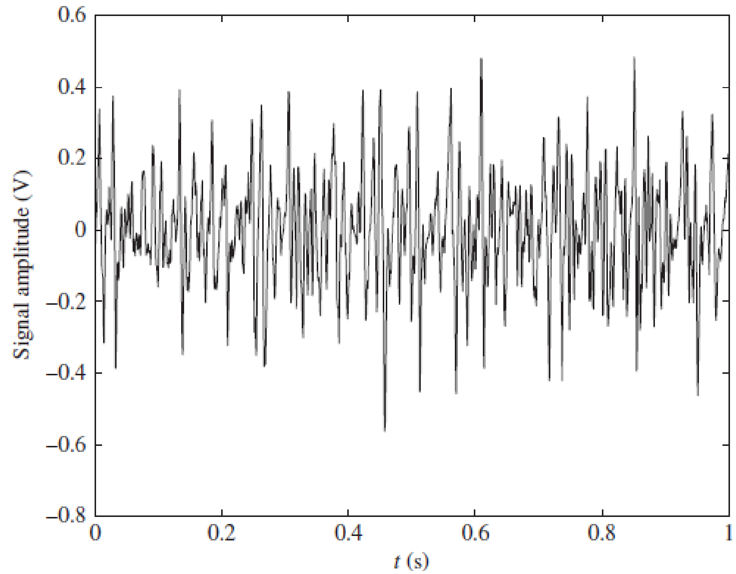
- 1777 –1855
- A German mathematician

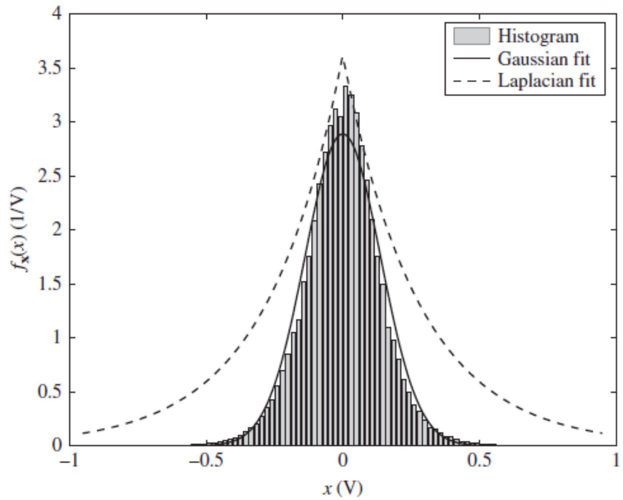
Ex. Muscle Activity

- Look at electrical activity of skeletal muscle by recording a human electromyogram (EMG).



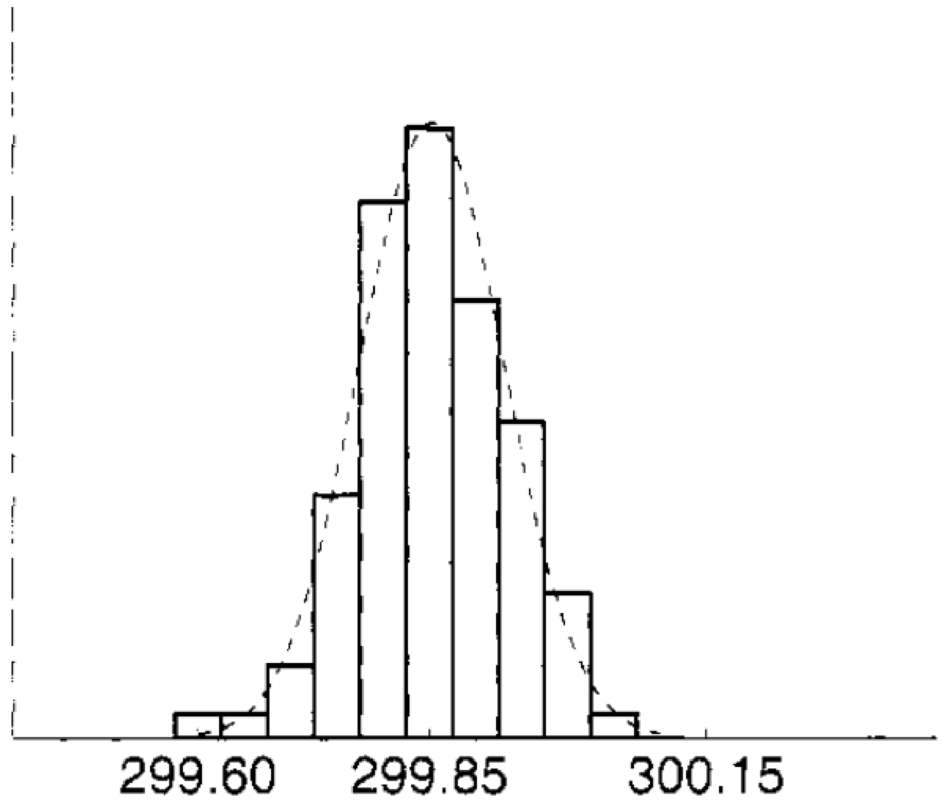
[http://www.adinstruments.com/solutions/education/ltexp/electromyography-emg-german]





Ex. Measuring the speed of light

- 100 measurements of the speed of light (×1,000 km/second), conducted by Albert Abraham Michelson in 1879.

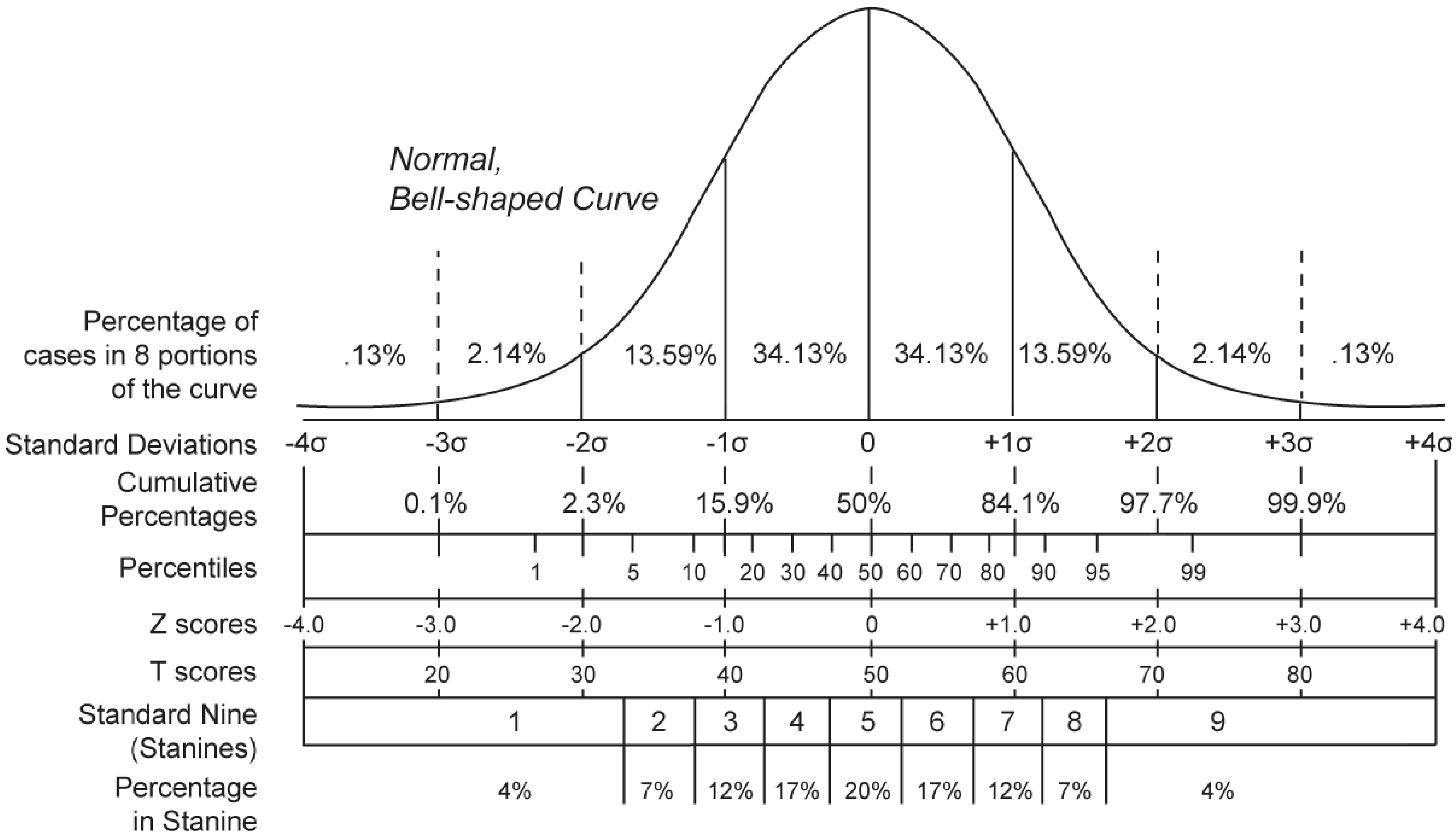


Expected Value and Variance

"Proof" by MATLAB's symbolic calculation

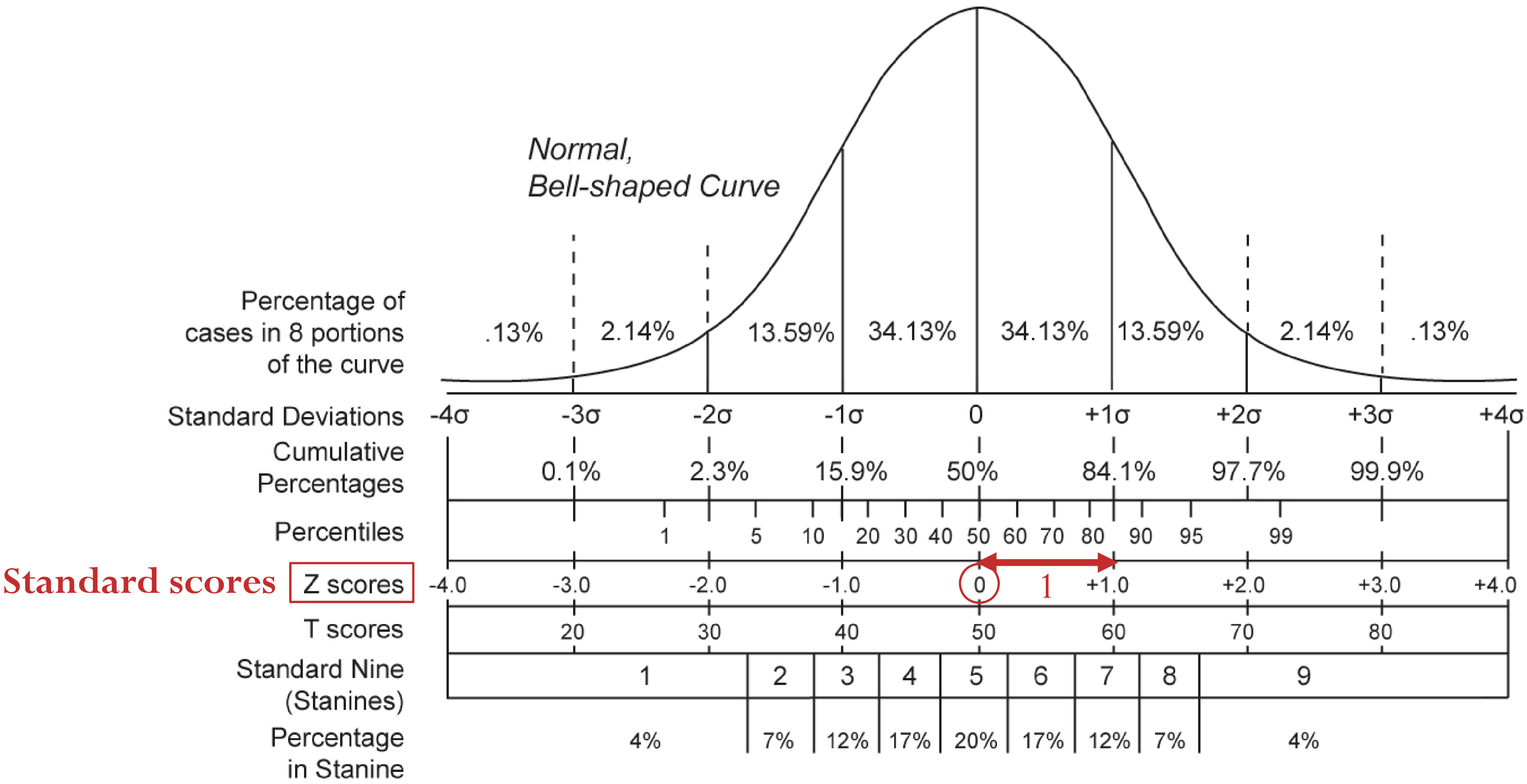
```
>> syms x
>> syms m real
>> syms sigma positive
>> int(1/(sqrt(sym(2)*pi)*sigma)*exp(-(x-m)^2/(2*sigma^2)),x,-inf,inf)
ans =
1
>> EX = int(x/(sqrt(sym(2)*pi)*sigma)*exp(-(x-m)^2/(2*sigma^2)),x,-inf,inf)
EX =
m
>> EX2 = int(x^2/(sqrt(sym(2)*pi)*sigma)*exp(-(x-m)^2/(2*sigma^2)),x,-inf,inf)
EX2 =
-(2^(1/2)*(limit(- x*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) - x^2/(2*sigma^2)) - m*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) - x^2/(2*sigma^2)) -
(2^(1/2)*pi^(1/2)*sigma*erfi((2^(1/2)*(x - m)*i)/(2*sigma))*(m^2 + sigma^2)*i)/2, x == -Inf) - limit(- x*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) -
x^2/(2*sigma^2)) - m*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) - x^2/(2*sigma^2)) - (2^(1/2)*pi^(1/2)*sigma*erfi((2^(1/2)*(x - m)*i)/(2*sigma))*(m^2 +
sigma^2)*i)/2, x == Inf)))/(2*pi^(1/2)*sigma)
>> EX2 = simplify(EX2)
EX2 =
m^2 + sigma^2
>> VarX = EX2 - (EX)^2
VarX =
sigma^2
```

Gaussian Random Variable



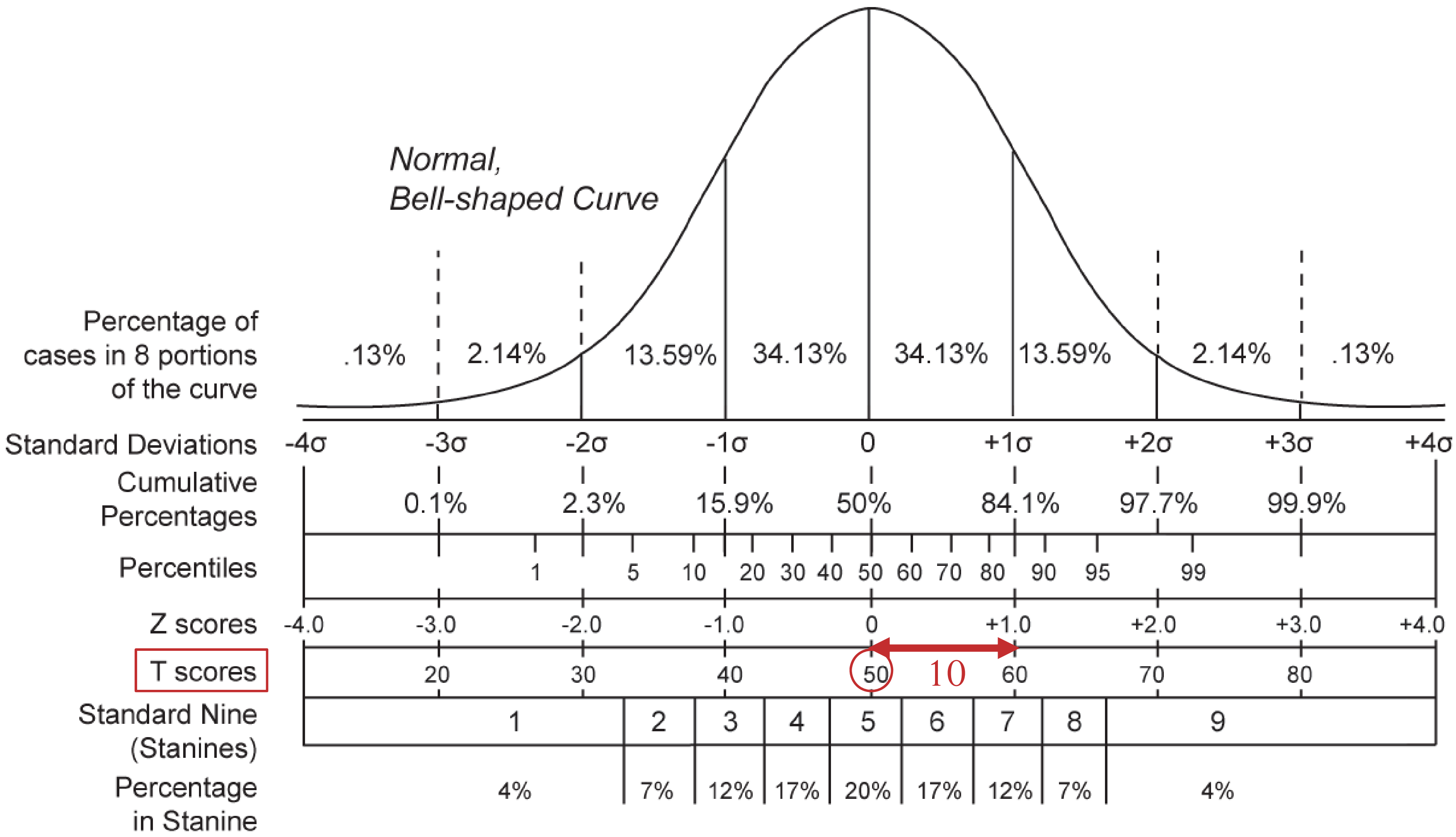
[Wikipedia.org]

Gaussian Random Variable



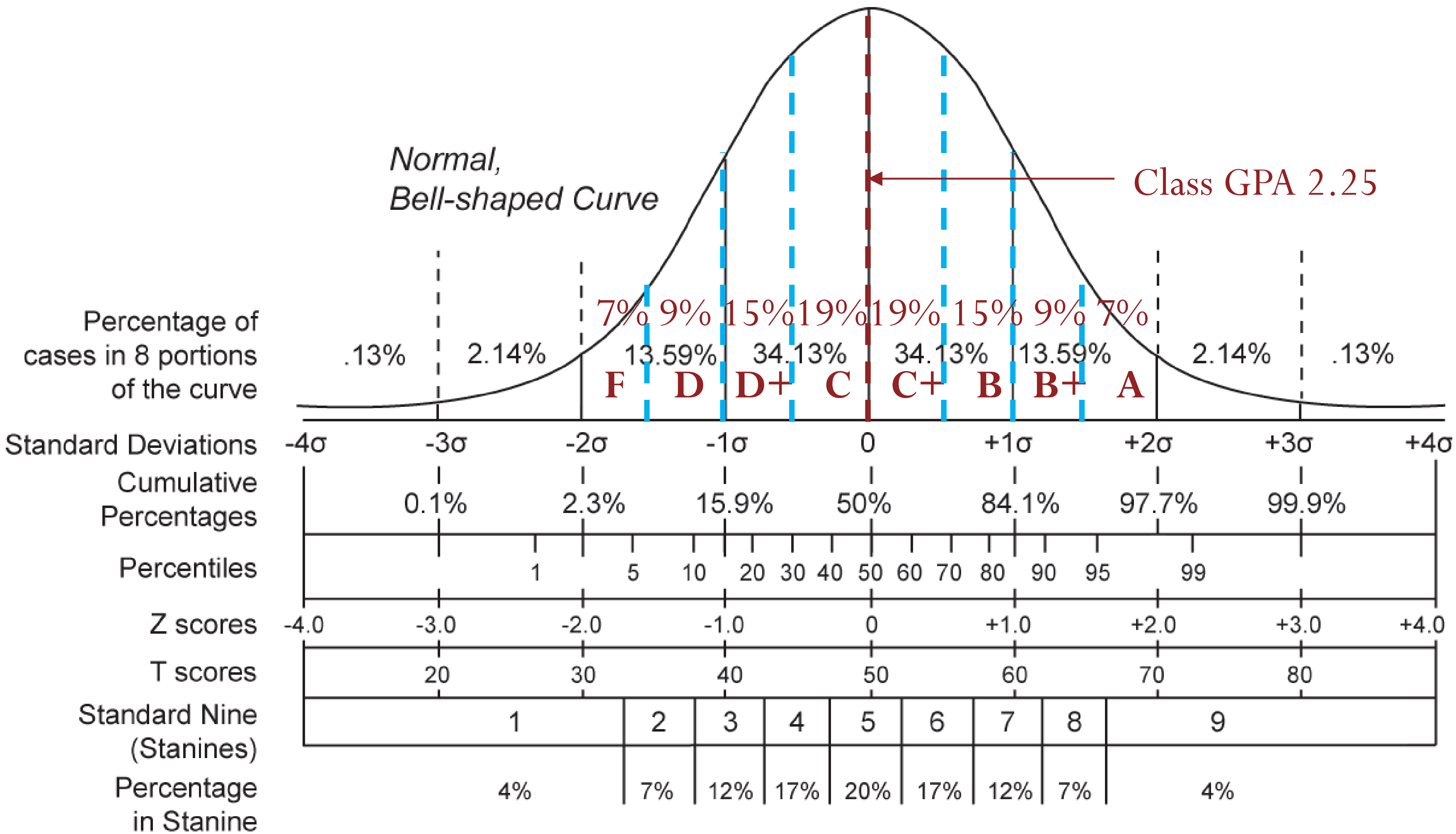
[Wikipedia.org]

Gaussian Random Variable



[Wikipedia.org]

SIIT Grading Scheme (Option 3)



[Wikipedia.org]

From the News

Higgs boson-like particle discovery claimed at LHC

COMMENTS (1665)

By Paul Rincon
Science editor, BBC News website, Geneva

4 July 2012

$$\frac{1}{1-\Phi(5)} \approx 3.5\times 10^6$$

$$\frac{1}{1-\Phi(4.9)} \approx 2\times 10^6$$

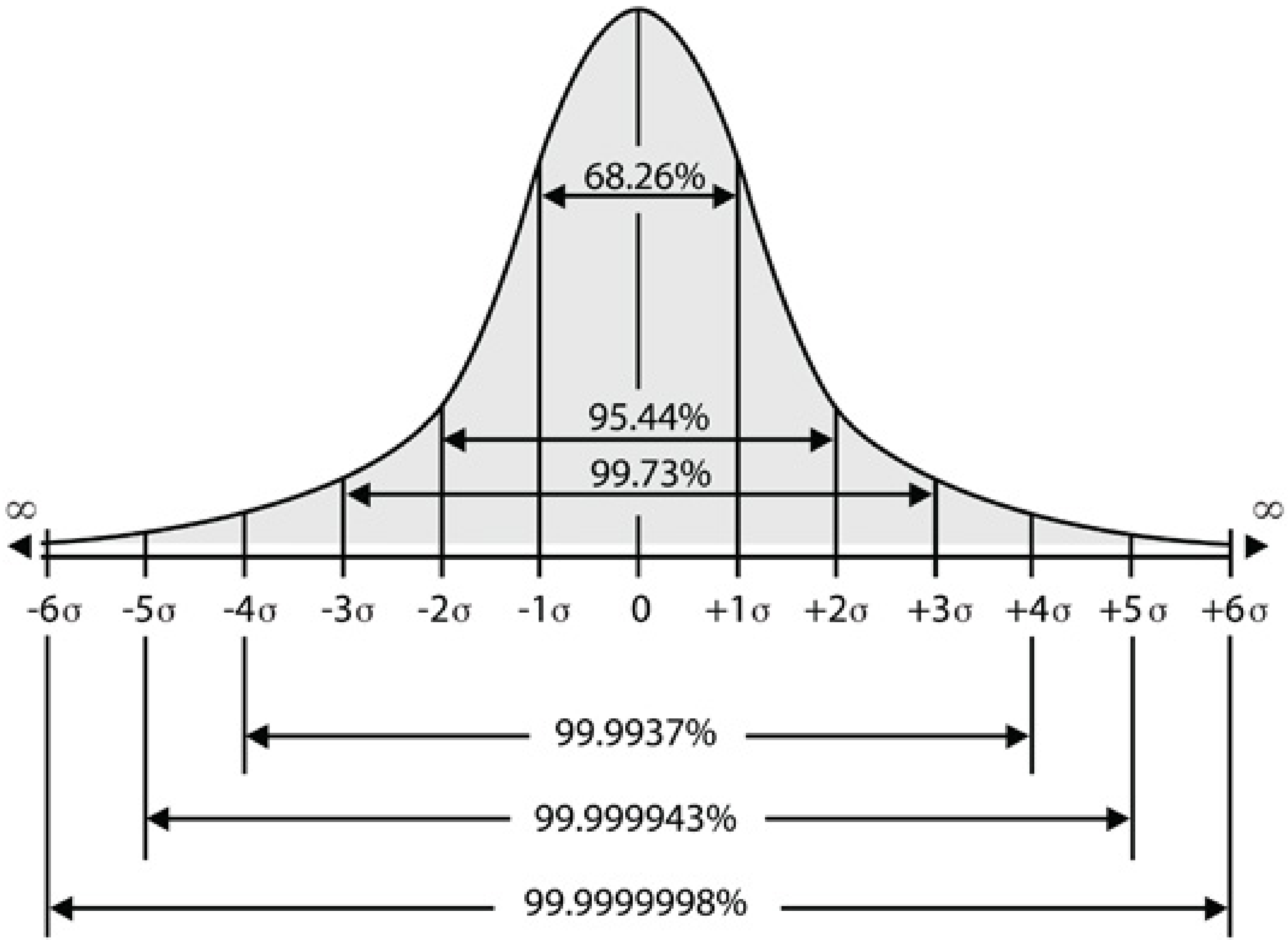
Particle physics has an accepted definition for a **discovery**: a "five-sigma" (or five standard-deviation) level of certainty

The number of sigmas measures how unlikely it is to get a certain experimental result as a matter of chance rather than due to a real effect

They claimed that by combining two data sets, they had attained a confidence level just at the "five-sigma" point - about a **one-in-3.5 million chance** that the signal they see would appear if there were no Higgs particle.

However, a full combination of the CMS data brings that number just back to **4.9 sigma** - a one-in-two million chance.

Six Sigma



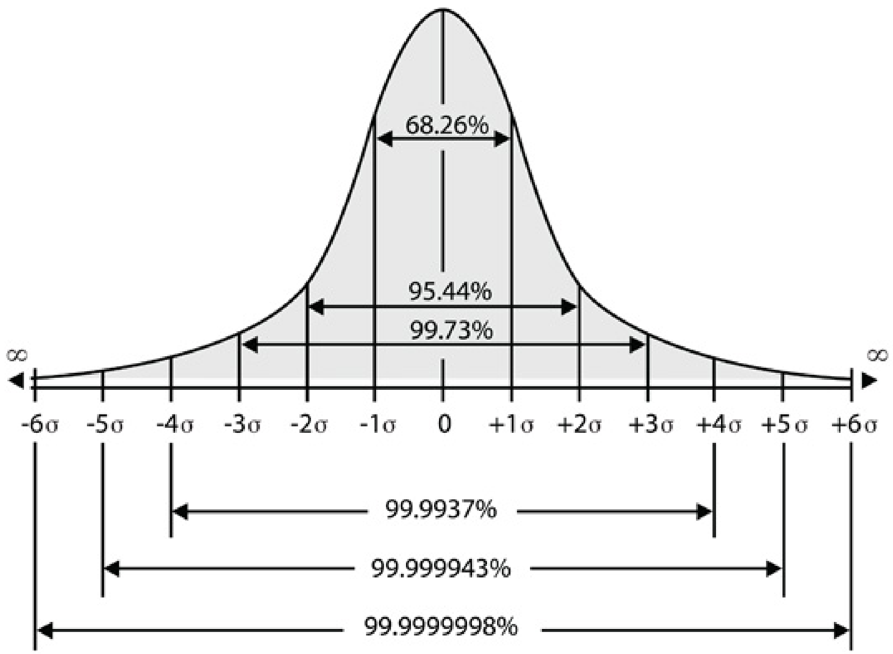
68.26%
95.44%
99.73%
∞
∞
-6σ
-5σ
-4σ
-3σ
-2σ
-1σ
0
+1σ
+2σ
+3σ
+4σ
+5σ
+6σ
99.9937%
99.999943%
99.9999998%

Six Sigma

- If you **manufacture** something that has a normal distribution and get an observation outside six σ of μ, you have either seen something extremely unlikely or there is something wrong with your manufacturing process. You'd better look it over.
- This approach is an example of **statistical quality control**, which has been used extensively and saved companies a lot of money in the last couple of decades.
- The term **Six Sigma**, a registered trademark of **Motorola**, has evolved to denote a methodology to monitor, control, and improve products and processes.
- There are Six Sigma societies, institutes, and conferences.
- Whatever Six Sigma has grown into, it all started with considerations regarding the normal distribution.

[Olofsson, 2006, p. 168]

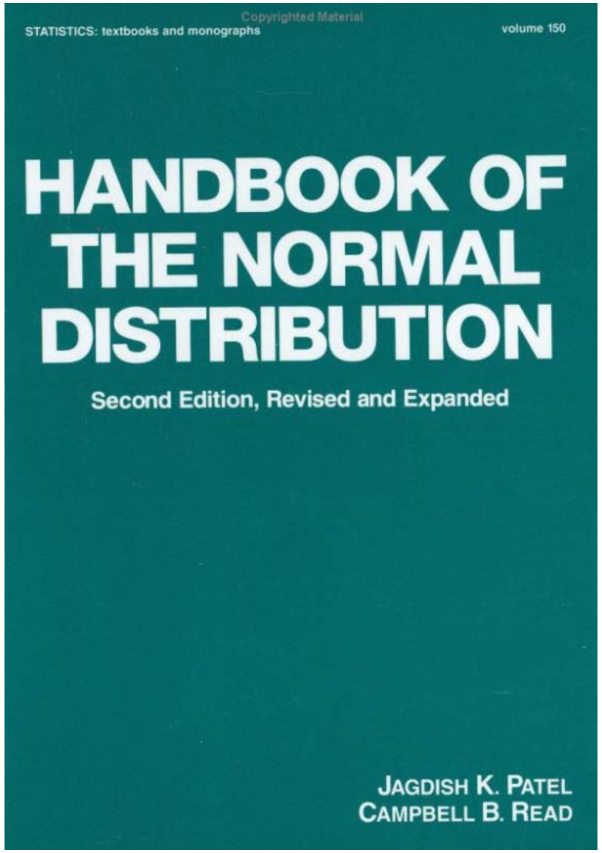
Six Sigma

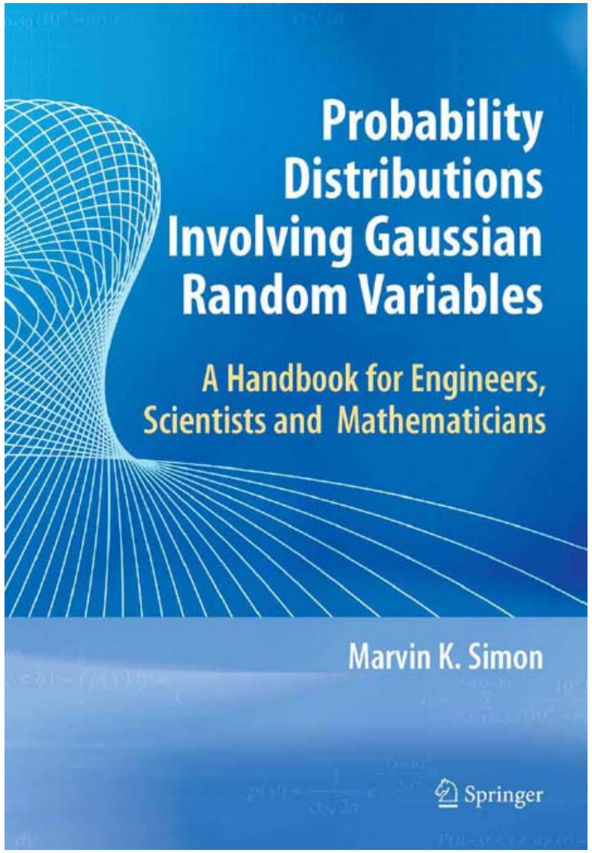


Range around μ	Percentage of products in conformance	Percentage of nonconforming products
-1σ to $+1\sigma$	68.26	31.74
-2σ to $+2\sigma$	95.46	4.54
-3σ to $+3\sigma$	99.73	0.27
-4σ to $+4\sigma$	99.9937	0.0063
-5σ to $+5\sigma$	99.999943	0.000057
-6σ to $+6\sigma$	99.9999998	0.00000002

[Bass, 2007, p. 20]

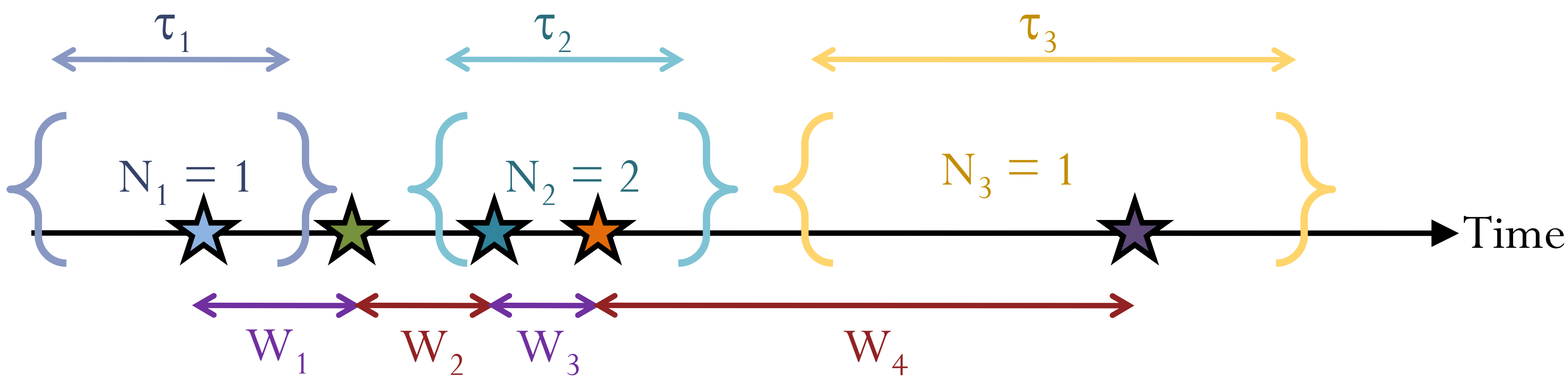
More on Gaussian RVs…





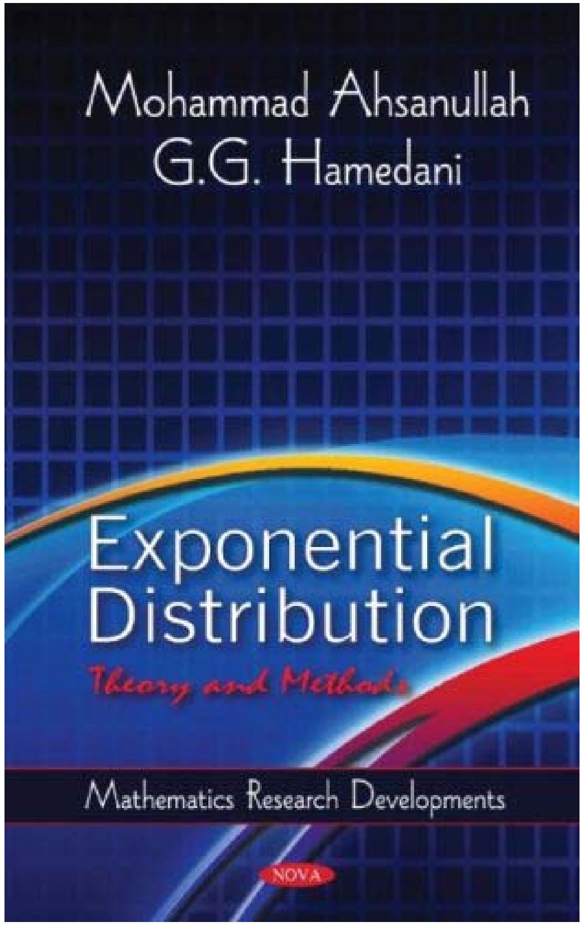
Poisson Process

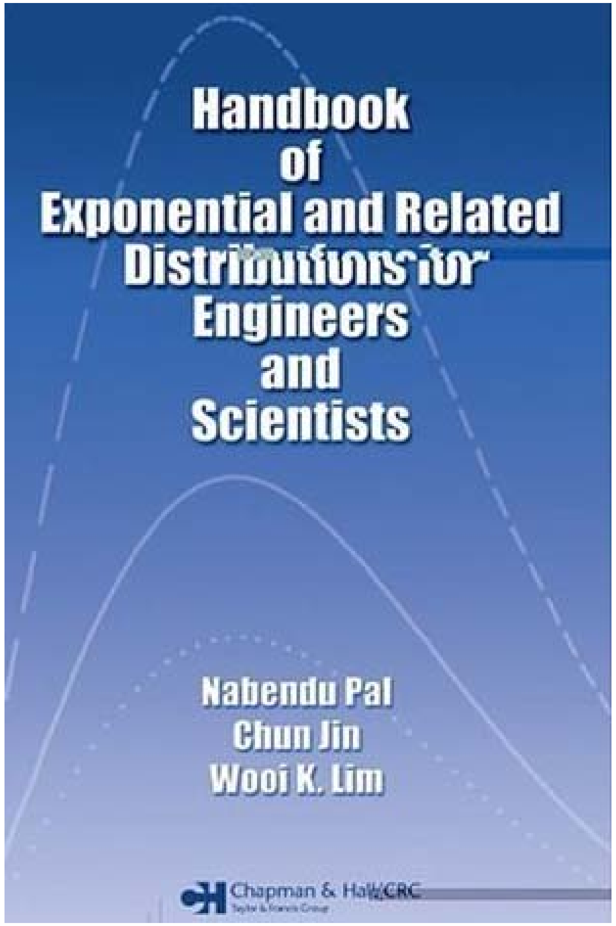
The number of arrivals $N_1, N_2, N_3, \ldots$ during non-overlapping time intervals are independent Poisson random variables with mean $= \lambda \times$ the length of the corresponding interval.



The lengths of time between adjacent arrivals $W_1, W_2, W_3, \ldots$ are i.i.d. exponential random variables with mean $1/\lambda$.

More on Exponential RV …





Review: Function of discrete RV

Example 9.16. Let

$$p_X(x) = \begin{cases} \frac{1}{c}x^2, & x = \pm 1, \pm 2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$Y = X^4.$$

Find $p_Y(y)$ and then calculate $\mathbb{E}Y$.

Step 1: Find c

$$\sum_x p_X(x) = 1$$

$$\frac{1}{c} + \frac{1}{c} + \frac{4}{c} + \frac{4}{c} = 1$$

$$\frac{1}{c}(10) = 1$$

$$c = 10$$

Step 2: Find $p_Y(y)$

$p_X(x)$	x	$y = x^4$
$1/c$	1	1
$1/c$	-1	1
$4/c$	2	16
$4/c$	-2	16

$$p_Y(1) = P[Y=1] = \frac{1}{c} + \frac{1}{c} = \frac{2}{c} = \frac{2}{10} = \frac{1}{5}$$

$$p_Y(16) = P[Y=16] = \frac{4}{c} + \frac{4}{c} = \frac{8}{c} = \frac{8}{10} = \frac{4}{5}$$

$$p_Y(y) = \begin{cases} 1/5, & y = 1, \\ 4/5, & y = 16, \\ 0, & \text{otherwise.} \end{cases}$$

Step 3:

$$\mathbb{E}Y = \sum_y y\, p_Y(y)$$

$$= 1 \times \frac{1}{5} + 16 \times \frac{4}{5}$$

$$= \frac{1 + 64}{5} = \frac{65}{5} = 13$$

References

- From Discrete to Continuous Random Variables: [Y&G] Sections 3.0 to 3.1
- PDF and CDF: [Y&G] Sections 3.1 to 3.2
- Expectation and Variance: [Y&G] Section 3.3
- Families of Continuous Random Variables: [Y&G] Sections 3.4 to 3.5
- SISO: [Y&G] Section 3.7; [Z&T] Section 5.2.5