ECS 315: Probability and Random Processes **2014/1**

HW Solution 6 — Due: Not Due

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Problem 1. The thickness of the wood paneling (in inches) that a customer orders is a random variable with the following cdf:

$$F_X(x) = \begin{cases} 0, & x < \frac{1}{8} \\ 0.2, & \frac{1}{8} \le x < \frac{1}{4} \\ 0.9, & \frac{1}{4} \le x < \frac{3}{8} \\ 1 & x \ge \frac{3}{8} \end{cases}$$

Determine the following probabilities:

(a) $P[X \le 1/18]$

(b) $P[X \le 1/4]$

(c) $P[X \le 5/16]$

(d) $P[X > 1/4]$

(e) $P[X \le 1/2]$

[Montgomery and Runger, 2010, Q3-42]

Solution:

(a) $P[X \le 1/18] = F_X(1/18) = \boxed{0}$ because $\frac{1}{18} < \frac{1}{8}$.

(b) $P[X \le 1/4] = F_X(1/4) = \boxed{0.9}$.

(c) $P[X \le 5/16] = F_X(5/16) = \boxed{0.9}$ because $\frac{1}{4} < \frac{5}{16} < \frac{3}{8}$.

(d) $P[X > 1/4] = 1 - P[X \le 1/4] = 1 - F_X(1/4) = 1 - 0.9 = \boxed{0.1}$.

(e) $P[X \le 1/2] = F_X(1/2) = \boxed{1}$ because $\frac{1}{2} > \frac{3}{8}$.

Alternatively, we can also derive the pmf first and then calculate the probabilities.

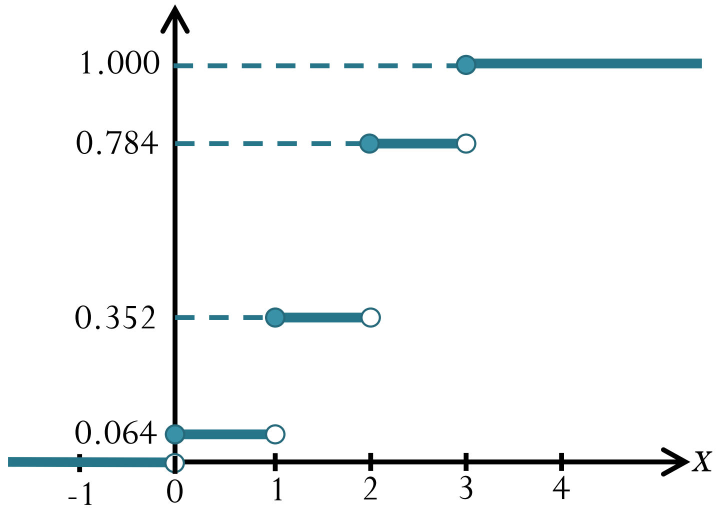

Figure 6.1: CDF of X for Problem 2

Problem 2. [M2011/1] The cdf of a random variable X is plotted in Figure 6.1.

(a) Find the pmf $p_X(x)$.

(b) Find the family to which X belongs. (Uniform, Bernoulli, Binomial, Geometric, Poisson, etc.)

Solution:

(a) For discrete random variable, $P[X = x]$ is the jump size at x on the cdf plot. In this problem, there are four jumps at 0, 1, 2, 3.

- $P[X = 0]$ = the jump size at $0 = 0.064 = \frac{64}{1000} = (4/10)^3 = (2/5)^3$.
- $P[X = 1]$ = the jump size at $1 = 0.352 - 0.064 = 0.288$.
- $P[X = 2]$ = the jump size at $2 = 0.784 - 0.352 = 0.432$.
- $P[X = 3]$ = the jump size at $3 = 1 - 0.784 = 0.216 = (6/10)^3$.

In conclusion,

$$\boxed{p_X(x) = \begin{cases} 0.064, & x = 0, \\ 0.288, & x = 1, \\ 0.432, & x = 2, \\ 0.216, & x = 3, \\ 0, & \text{otherwise.} \end{cases}}$$

(b) Among all the pmf that we discussed in class, only one can have support $= \{0, 1, 2, 3\}$ with unequal probabilities. This is the $\boxed{\text{binomial}}$ pmf. To check that it really is Binomial, recall that the pmf for binomial X is given by $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$ for $x = 0, 1, 2, \ldots, n$. Here, $n = 3$. Furthermore, observe that $p_X(0) = (1-p)^n$. By comparing $p_X(0)$ with what we had in part (a), we have $1 - p = 2/5$ or $p = 3/5$. For $x = 1, 2, 3$, plugging in $p = 3/5$ and $n = 3$ in to $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$ gives the same values as what we had in part (a). So, X is a binomial RV.

Problem 3. Arrivals of customers at the local supermarket are modeled by a Poisson process with a rate of $\lambda = 2$ customers per minute. Let M be the number of customers arriving between 9:00 and 9:05. What is the probability that $M < 2$?

Solution: Here, we are given that $M \sim \mathcal{P}(\alpha)$ where $\alpha = \lambda T = 2 \times 5 = 10$. Recall that, for $M \sim \mathcal{P}(\alpha)$, we have

$$P[M = m] = \begin{cases} e^{-\alpha} \frac{\alpha^m}{m!}, & m \in \{0, 1, 2, 3, \ldots\} \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} P[M < 2] &= P[M = 0] + P[M = 1] = e^{-\alpha} \frac{\alpha^0}{0!} + e^{-\alpha} \frac{\alpha^1}{1!} \\ &= e^{-\alpha}(1 + \alpha) = e^{-10}(1 + 10) = 11e^{-10} \approx 5 \times 10^{-4}. \end{aligned}$$

Problem 4. When n is large, binomial distribution Binomial(n, p) becomes difficult to compute directly because of the need to calculate factorial terms. In this question, we will consider an approximation when the value of p is close to 0. In such case, the binomial can be approximated[1] by the Poisson distribution with parameter $\alpha = np$. For this approximation to work, we will see in this exercise that n does not have to be very large and p does not need to be very small.

(a) Let $X \sim$ Binomial$(12, 1/36)$. (For example, roll two dice 12 times and let X be the number of times a double 6 appears.) Evaluate $p_X(x)$ for $x = 0, 1, 2$.

(b) Compare your answers part (a) with its Poisson approximation.

[1]More specifically, suppose X_n has a binomial distribution with parameters n and p_n. If $p_n \to 0$ and $np_n \to \alpha$ as $n \to \infty$, then

$$P[X_n = k] \to e^{-\alpha} \frac{\alpha^k}{k!}.$$

(c) Compare `MATLAB` plots of $p_X(x)$ in part (a) and the pmf of $\mathcal{P}(np)$.

Solution:

(a) For Binomial(n,p) random variable,

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0,1,2,\dots,n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we are given that $n = 12$ and $p = \frac{1}{36}$. Plugging in $x = 0,1,2$, we get $\boxed{0.7132, 0.2445, 0.0384}$, respectively

(b) A Poisson random variable with parameter $\alpha = np$ can approximate a Binomial(n,p) random variable when n is large and p is small. Here, with $n = 12$ and $p = \frac{1}{36}$, we have $\alpha = 12 \times \frac{1}{36} = \frac{1}{3}$. The Poisson pmf at $x = 0,1,2$ is given by $e^{-\alpha}\frac{\alpha^x}{x!} = e^{-1/3}\frac{(1/3)^x}{x!}$. Plugging in $x = 0,1,2$ gives $\boxed{0.7165, 0.2388, 0.0398}$, respectively.

(c) See Figure 6.2. Note how close they are!

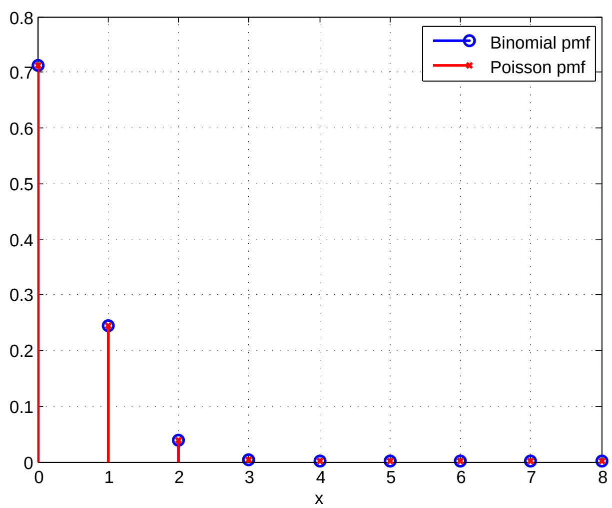

Figure 6.2: Poisson Approximation

Problem 5. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson pmf. (For simplicity, exclude birthdays on February 29.) [Bertsekas and Tsitsiklis, 2008, Q2.2.2]

Solution: Let N be the number of guests that has the same birthday as you. We may think of the comparison of your birthday with each of the guests as a Bernoulli trial. Here, there are 500 guests and therefore we are considering $n = 500$ trials. For each trial, the (success) probability that you have the same birthday as the corresponding guest is $p = \frac{1}{365}$. Then, this $N \sim \text{Binomial}(n, p)$.

(a) Binomial: $P[N = 1] = np^1(1-p)^{n-1} \approx 0.348$.

(b) Poisson: $P[N = 1] = e^{-np}\frac{(np)^1}{1!} \approx 0.348$.

Extra Questions

Here are some questions for those who want extra practice.

Problem 6. A sample of a radioactive material emits particles at a rate of 0.7 per second. Assuming that these are emitted in accordance with a Poisson distribution, find the probability that in one second

(a) exactly one is emitted,

(b) more than three are emitted,

(c) between one and four (inclusive) are emitted

[Applebaum, 2008, Q5.27].

Solution: Let X be the number or particles emitted during the one second under consideration. Then $X \sim \mathcal{P}(\alpha)$ where $\alpha = \lambda T = 0.7 \times 1 = 0.7$.

(a) $P[X = 1] = e^{-\alpha}\frac{\alpha^1}{1!} = \alpha e^{-\alpha} = 0.7e^{-0.7} \approx 0.3477$.

(b) $P[X > 3] = 1 - P[X \le 3] = 1 - \sum_{k=0}^{3} e^{-0.7}\frac{0.7^k}{k!} \approx 0.0058$.

(c) $P[1 \le X \le 4] = \sum_{k=1}^{4} e^{-0.7}\frac{0.7^k}{k!} \approx 0.5026$.

Problem 7 (M2011/1)**.** You are given an unfair coin with probability of obtaining a head equal to $1/3,000,000,000$. You toss this coin 6,000,000,000 times. Let A be the event that you get "tails for all the tosses". Let B be the event that you get "heads for all the tosses".

(a) Approximate $P(A)$.

(b) Approximate $P(A \cup B)$.

Solution: Let N be the number of heads among the n tosses. Then, $N \sim \mathcal{B}(n,p)$. Here, we have small $p = 1/3 \times 10^9$ and large $n = 6 \times 10^9$. So, we can apply Poisson approximation. In other words, $\mathcal{B}(n,p)$ is well-approximated by $\mathcal{P}(\alpha)$ where $\alpha = np = 2$.

(a) $P(A) = P[N=0] = e^{-2}\frac{2^0}{0!} = \frac{1}{e^2} \approx 0.1353$.

(b) $P(A \cup B) = P[N=0] + P[N=n] = e^{-2}\frac{2^0}{0!} + e^{-2}\frac{2^{6\times10^9}}{(6\times10^9)!}$. The second term is extremely small compared to the first one. Hence, $P(A \cup B)$ is approximately the same as $P(A)$.

Problem 8. In one of the New York state lottery games, a number is chosen at random between 0 and 999. Suppose you play this game 250 times. Use the Poisson approximation to estimate the probability that you will never win and compare this with the exact answer.

Solution:[Durrett, 2009, Q2.41] Let W be the number of wins. Then, $W \sim \text{Binomial}(250, p)$ where $p = 1/1000$. Hence,

$$P[W=0] = \binom{250}{0} p^0 (1-p)^{250} \approx 0.7787.$$

If we approximate W by $\Lambda \sim \mathcal{P}(\alpha)$. Then we need to set

$$\alpha = np = \frac{250}{1000} = \frac{1}{4}.$$

In which case,

$$P[\Lambda = 0] = e^{-\alpha}\frac{\alpha^0}{0!} = e^{-\alpha} \approx 0.7788$$

which is very close to the answer from direct calculation.