

## HW Solution 5 — Due: Sep 25

Lecturer: Prapun Suksompong, Ph.D.

**Instructions**

- (a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.
- (b) It is important that you try to solve all problems. (5 pt)  
The extra question at the end is optional.
- (c) Late submission will be heavily penalized.
- (d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

**Problem 1** (Majority Voting in Digital Communication). A certain binary communication system has a bit-error rate of 0.1; i.e., in transmitting a single bit, the probability of receiving the bit in error is 0.1. To transmit messages, a three-bit repetition code is used. In other words, to send the message 1, a “codeword” 111 is transmitted, and to send the message 0, a “codeword” 000 is transmitted. At the receiver, if two or more 1s are received, the decoder decides that message 1 was sent; otherwise, i.e., if two or more zeros are received, it decides that message 0 was sent.

Assuming bit errors occur independently, find the probability that the decoder puts out the wrong message.

[Gubner, 2006, Q2.62]

**Solution:** Let  $p = 0.1$  be the bit error rate. Let  $\mathcal{E}$  be the error event. (This is the event that the decoded bit value is not the same as the transmitted bit value.) Because majority voting is used, event  $\mathcal{E}$  occurs if and only if there are at least two bit errors. Therefore

$$P(\mathcal{E}) = \binom{3}{2}p^2(1-p) + \binom{3}{3}p^3 = p^2(3-2p).$$

When  $p = 0.1$ , we have  $P(\mathcal{E}) \approx \boxed{0.028}$ .

**Problem 2.** An optical inspection system is to distinguish among different part types. The probability of a correct classification of any part is 0.98. Suppose that three parts are inspected and that the classifications are independent.

- (a) Let the random variable  $X$  denote the number of parts that are correctly classified. Determine the probability mass function of  $X$ . [Montgomery and Runger, 2010, Q3-20]
- (b) Let the random variable  $Y$  denote the number of parts that are incorrectly classified. Determine the probability mass function of  $Y$ .

**Solution**<sup>1</sup>:

We will reexpress the problem in terms of Bernoulli trials so that we can use the results discussed in class. In this problem, we have three Bernoulli trials. Each trial deals with classification.

- (a) To find  $p_X(x)$ , first we find its support. Three parts are inspected here. Therefore,  $X$  can be 0, 1, 2, or 3. So, we need to find  $p_X(x) = P[X = x]$  when  $x = 0, 1, 2$  or 3. The pmf  $p_X(x)$  for other  $x$  values are all 0 because  $X$  cannot take the value of those  $x$ .

For each  $x \in \{0, 1, 2, 3\}$ ,  $p_X(x) = P[X = x]$  is simply the probability that exactly  $x$  parts are correctly classified. Note that, because we are interested in the correctly classified part, we define the “success” event for a trial to be the event that the part is classified correctly. We are given that the probability of a correct classification of any part is 0.98. Therefore, for each of our Bernoulli trials, the probability of success is  $p = 0.98$ . Under such interpretation (of “success”),  $p_X(x)$  is then the same as finding the probability of having exactly  $x$  successes in  $n = 3$  Bernoulli trials. We have seen in class that the probability of this is  $\binom{3}{x}p^x(1-p)^{3-x}$ . Plugging in  $p = 0.98$ , we have  $p_X(x) = \binom{3}{x}0.98^x(0.02)^{3-x}$  for  $x \in \{0, 1, 2, 3\}$ .

Combining the expression above with the cases for other  $x$  values, we then have

$$p_X(x) = \begin{cases} \binom{3}{x}0.98^x(0.02)^{3-x}, & x \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise} \end{cases} \quad (5.1)$$

In particular,  $p_X(0) = 8 \times 10^{-6}$ ,  $p_X(1) = 0.001176$ ,  $p_X(2) = 0.057624$ , and  $p_X(3) = 0.941192$ . In fact, this  $X$  is a binomial random variable with  $n = 3$  and  $p = 0.98$ . In **MATLAB**, the probabilities above can be calculated via the command `binopdf(0:3,3,0.98)`.

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<sup>1</sup>The solution provided here assumes that we still haven’t reached the part of the course where binomial random variable is discussed. Therefore, the pmf is derived by relying on the concept of Bernoulli trials and the formula discussed back when we studied that topic.

- (b) **Method 1:** Similar analysis is performed on the random variable  $Y$ . The only difference here is that, now, we are interested in the number of parts that are incorrectly classified. Therefore, we will define the “success” event for a trial to be the event that the part is classified incorrectly. We are given that the probability of a correct classification of any part is 0.98. Therefore, for each of our Bernoulli trials, the probability of success is  $1 - p = 1 - 0.98 = 0.02$ . With this new probability of success, we have

$$p_Y(y) = \begin{cases} \binom{3}{y} 0.02^y (0.98)^{3-y}, & y \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise} \end{cases} \quad (5.2)$$

In particular,  $p_Y(0) = 0.941192$ ,  $p_Y(1) = 0.057624$ ,  $p_Y(2) = 0.001176$ , and  $p_Y(3) = 8 \times 10^{-6}$ . In fact, this  $Y$  is a binomial random variable with  $n = 3$  and  $p = 0.02$ . In MATLAB, the probability values above can be calculated via the command `binopdf(0:3,3,0.02)`.

**Method 2:** Alternatively, note that there are three parts. If  $X$  of them are classified correctly, then the number of incorrectly classified parts is  $n - X$ , which is what we defined as  $Y$ . Therefore,  $Y = 3 - X$ . Hence,  $p_Y(y) = P[Y = y] = P[3 - X = y] = P[X = 3 - y] = p_X(3 - y)$ .

**Problem 3.** Consider the sample space  $\Omega = \{-2, -1, 0, 1, 2, 3, 4\}$ . Suppose that  $P(A) = |A|/|\Omega|$  for any event  $A \subset \Omega$ . Define the random variable  $X(\omega) = \omega^2$ . Find the probability mass function of  $X$ .

**Solution:** The random variable maps the outcomes  $\omega = -2, -1, 0, 1, 2, 3, 4$  to numbers  $x = 4, 1, 0, 1, 4, 9, 16$ , respectively. Therefore,

$$\begin{aligned} p_X(0) &= P(\{\omega : X(\omega) = 0\}) = P(\{0\}) = \frac{1}{7}, \\ p_X(1) &= P(\{\omega : X(\omega) = 1\}) = P(\{-1, 1\}) = \frac{2}{7}, \\ p_X(4) &= P(\{\omega : X(\omega) = 4\}) = P(\{-2, 2\}) = \frac{2}{7}, \\ p_X(9) &= P(\{\omega : X(\omega) = 9\}) = P(\{3\}) = \frac{1}{7}, \text{ and} \\ p_X(16) &= P(\{\omega : X(\omega) = 16\}) = P(\{4\}) = \frac{1}{7}. \end{aligned}$$

Combining the results above, we get the complete pmf:

$$p_X(x) = \begin{cases} \frac{1}{7}, & x = 0, 9, 16, \\ \frac{2}{7}, & x = 1, 4, \\ 0, & \text{otherwise.} \end{cases}$$

**Problem 4.** Suppose  $X$  is a random variable whose pmf at  $x = 0, 1, 2, 3, 4$  is given by  $p_X(x) = \frac{2x+1}{25}$ .

Remark: Note that the statement above does not specify the value of the  $p_X(x)$  at the value of  $x$  that is not 0,1,2,3, or 4.

- (a) What is  $p_X(5)$ ?
- (b) Determine the following probabilities:
- (i)  $P[X = 4]$
  - (ii)  $P[X \leq 1]$
  - (iii)  $P[2 \leq X < 4]$
  - (iv)  $P[X > -10]$

**Solution:**

- (a) First, we calculate

$$\sum_{x=0}^4 p_X(x) = \sum_{x=0}^4 \frac{2x+1}{25} = \frac{25}{25} = 1.$$

Therefore, there can't be any other  $x$  with  $p_X(x) > 0$ . At  $x = 5$ , we then conclude that  $p_X(5) = \boxed{0}$ . The same reasoning also implies that  $p_X(x) = 0$  at any  $x$  that is not 0,1,2,3, or 4.

- (b) Recall that, for discrete random variable  $X$ , the probability

$$P[\text{some condition(s) on } X]$$

can be calculated by adding  $p_X(x)$  for all  $x$  in the support of  $X$  that satisfies the given condition(s).

- (i)  $P[X = 4] = p_X(4) = \frac{2 \times 4 + 1}{25} = \boxed{\frac{9}{25}}$ .
- (ii)  $P[X \leq 1] = p_X(0) + p_X(1) = \frac{2 \times 0 + 1}{25} + \frac{2 \times 1 + 1}{25} = \frac{1}{25} + \frac{3}{25} = \boxed{\frac{4}{25}}$ .
- (iii)  $P[2 \leq X < 4] = p_X(2) + p_X(3) = \frac{2 \times 2 + 1}{25} + \frac{2 \times 3 + 1}{25} = \frac{5}{25} + \frac{7}{25} = \boxed{\frac{12}{25}}$ .
- (iv)  $P[X > -10] = \boxed{1}$  because all the  $x$  in the support of  $X$  satisfies  $x > -10$ .

**Problem 5.** The random variable  $V$  has pmf

$$p_V(v) = \begin{cases} cv^2, & v = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

- Find the value of the constant  $c$ .
- Find  $P[V \in \{u^2 : u = 1, 2, 3, \dots\}]$ .
- Find the probability that  $V$  is an even number.
- Find  $P[V > 2]$ .
- Sketch  $p_V(v)$ .
- Sketch  $F_V(v)$ . (Note that  $F_V(v) = P[V \leq v]$ .)

**Solution:** [Y&G, Q2.2.3]

- We choose  $c$  so that the pmf sums to one:

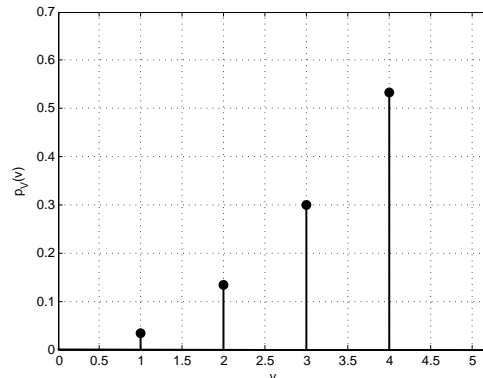
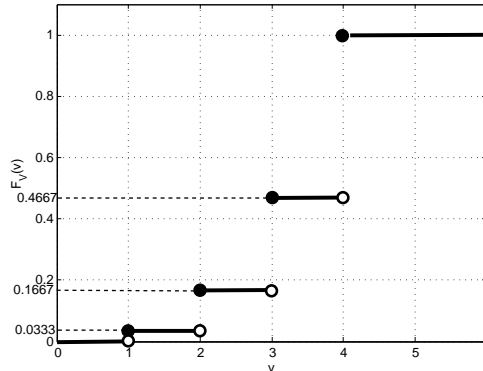
$$\sum_v p_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1.$$

Hence,  $c = \boxed{1/30}$ .

- $P[V \in \{u^2 : u = 1, 2, 3, \dots\}] = p_V(1) + p_V(4) = c(1^2 + 4^2) = \boxed{17/30}$ .
- $P[V \text{ even}] = p_V(2) + p_V(4) = c(2^2 + 4^2) = 20/30 = \boxed{2/3}$ .
- $P[V > 2] = p_V(3) + p_V(4) = c(3^2 + 4^2) = 25/30 = \boxed{5/6}$ .
- See Figure 5.1 for the sketch of  $p_V(v)$ :
- See Figure 5.2 for the sketch of  $F_V(v)$ :

## Extra Question

**Problem 6.** Consider a transmission over a binary symmetric channel (BSC) with crossover probability  $p$ . The random (binary) input to the BSC is denoted by  $X$ . Let  $p_1$  be the probability that  $X = 1$ . (This implies the probability that  $X = 0$  is  $1 - p_1$ .) Let  $Y$  be the output of the BSC.

Figure 5.1: Sketch of  $p_V(v)$  for Question 5Figure 5.2: Sketch of  $F_V(v)$  for Question 5

- (a) Suppose, at the receiver (which observes the output of the BSC), we learned that  $Y = 1$ . For each of the following scenarios, which event is more likely, “ $X = 1$  was transmitted” or “ $X = 0$  was transmitted”? (Hint: Use Bayes’ theorem.)
- Assume  $p = 0.3$  and  $p_1 = 0.1$ .
  - Assume  $p = 0.3$  and  $p_1 = 0.5$ .
  - Assume  $p = 0.3$  and  $p_1 = 0.9$ .
  - Assume  $p = 0.7$  and  $p_1 = 0.5$ .
- (b) Suppose, at the receiver (which observes the output of the BSC), we learned that  $Y = 0$ . For each of the following scenarios, which event is more likely, “ $X = 1$  was transmitted” or “ $X = 0$  was transmitted”?
- Assume  $p = 0.3$  and  $p_1 = 0.1$

- (ii) Assume  $p = 0.3$  and  $p_1 = 0.5$
- (iii) Assume  $p = 0.3$  and  $p_1 = 0.9$
- (iv) Assume  $p = 0.7$  and  $p_1 = 0.5$

Remark: A MAP (maximum a posteriori) detector is a detector that takes the observed value  $Y$  and then calculate the most likely transmitted value. More specifically,

$$\hat{x}_{MAP}(y) = \arg \max_x P[X = x | Y = y]$$

In fact, in part (a), each of your answers is  $\hat{x}_{MAP}(1)$  and in part (b), each of your answers is  $\hat{x}_{MAP}(0)$ .

**Solution:** First, recall that, in class, we define  $P[X = x]$  to be  $P([X = x])$ . Here, we extend the such definition to conditional probability. In particular,

$$P[Y = y | X = x] = P([Y = y] | [X = x]).$$

Here, we are given that  $P[X = 1] = p_1$ . Applying  $P(A^c) = 1 - P(A)$ , we have  $P[X = 0] = 1 - p_1$ . We are also given that

$$P[Y = 1 | X = 0] = P[Y = 0 | X = 1] = p.$$

Applying  $P(A^c | B) = 1 - P(A | B)$ , we have

$$P[Y = 0 | X = 0] = P[Y = 1 | X = 1] = 1 - p.$$

- (a) Here, we know that  $Y = 1$ . To find out what was transmitted, we compare  $P[X = 0 | Y = 1]$  and  $P[X = 1 | Y = 1]$ . By Bayes' theorem,

$$P[X = 0 | Y = 1] = \frac{P[Y = 1 | X = 0] P[X = 0]}{P[Y = 1]} = \frac{p(1 - p_1)}{P[Y = 1]} = \frac{p - pp_1}{P[Y = 1]}$$

and

$$P[X = 1 | Y = 1] = \frac{P[Y = 1 | X = 1] P[X = 1]}{P[Y = 1]} = \frac{(1 - p)p_1}{P[Y = 1]} = \frac{p_1 - pp_1}{P[Y = 1]}$$

Note that both terms have “ $-pp_1$ ” in the numerator and “ $P[Y = 1]$ ” the denominator. So, we can simply compare the “ $p$ ” and “ $p_1$ ” parts.

- (i) When  $p = 0.3$  and  $p_1 = 0.1$ , we have  $p > p_1$ . Therefore,  $P[X = 0 | Y = 1] > P[X = 1 | Y = 1]$ . In other words, conditioned on  $Y = 1$ , the event  $X = \boxed{0}$  is more likely.

- (ii) When  $p = 0.3$  and  $p_1 = 0.5$ , we have  $p < p_1$ . Therefore,  $P[X = 0 | Y = 1] < P[X = 1 | Y = 1]$ . In other words, conditioned on  $Y = 1$ , the event  $X = \boxed{1}$  is more likely.
- (iii) When  $p = 0.3$  and  $p_1 = 0.9$ , we have  $p < p_1$ . Therefore,  $P[X = 0 | Y = 1] < P[X = 1 | Y = 1]$ . In other words, conditioned on  $Y = 1$ , the event  $X = \boxed{1}$  is more likely.
- (iv) When  $p = 0.7$  and  $p_1 = 0.5$ , we have  $p > p_1$ . Therefore,  $P[X = 0 | Y = 1] > P[X = 1 | Y = 1]$ . In other words, conditioned on  $Y = 1$ , the event  $X = \boxed{0}$  is more likely.
- (b) In this part, we know that  $Y = 0$ . To find out what was transmitted, we compare  $P[X = 0 | Y = 0]$  and  $P[X = 1 | Y = 0]$ . By Bayes' theorem,

$$P[X = 0 | Y = 0] = \frac{P[Y = 0 | X = 0] P[X = 0]}{P[Y = 0]} = \frac{(1 - p)(1 - p_1)}{P[Y = 0]} = \frac{1 - p - p_1 + pp_1}{P[Y = 0]}$$

and

$$P[X = 1 | Y = 0] = \frac{P[Y = 0 | X = 1] P[X = 1]}{P[Y = 0]} = \frac{pp_1}{P[Y = 0]} = \frac{pp_1}{P[Y = 0]}$$

Note that both terms have “ $-pp_1$ ” in the numerator and “ $P[Y = 0]$ ” the denominator. So, we can simply compare the “ $1 - p - p_1$ ” and “ $0$ ” parts.

- (i) When  $p = 0.3$  and  $p_1 = 0.1$ , we have  $1 - p - p_1 = 0.6 > 0$ . Therefore,  $P[X = 0 | Y = 0] > P[X = 1 | Y = 0]$ . In other words, conditioned on  $Y = 0$ , the event  $X = \boxed{0}$  is more likely.
- (ii) When  $p = 0.3$  and  $p_1 = 0.5$ , we have  $1 - p - p_1 = 0.2 > 0$ . Therefore,  $P[X = 0 | Y = 0] > P[X = 1 | Y = 0]$ . In other words, conditioned on  $Y = 0$ , the event  $X = \boxed{0}$  is more likely.
- (iii) When  $p = 0.3$  and  $p_1 = 0.9$ , we have  $1 - p - p_1 = -0.2 < 0$ . Therefore,  $P[X = 0 | Y = 0] < P[X = 1 | Y = 0]$ . In other words, conditioned on  $Y = 1$ , the event  $X = \boxed{1}$  is more likely.
- (iv) When  $p = 0.7$  and  $p_1 = 0.5$ , we have  $1 - p - p_1 = -0.2 < 0$ . Therefore,  $P[X = 0 | Y = 0] < P[X = 1 | Y = 0]$ . In other words, conditioned on  $Y = 0$ , the event  $X = \boxed{1}$  is more likely.