ECS 315: Probability and Random Processes **2014/1**

HW Solution 4 — Due: September 18

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Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)
The extra questions at the end are optional.

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases. A specific design is randomly generated by the Web server when you visit the site. Let A denote the event that the design color is red and let B denote the event that the font size is not the smallest one.

(a) Use classical probability to evaluate $P(A)$, $P(B)$ and $P(A \cap B)$. Show that the two events A and B are independent by checking whether $P(A \cap B) = P(A)P(B)$.

(b) Using the values of $P(A)$ and $P(B)$ from the previous part and the fact that $A \perp\!\!\!\perp B$, calculate the following probabilities.

(i) $P(A \cup B)$

(ii) $P(A \cup B^c)$

(iii) $P(A^c \cup B^c)$

[Montgomery and Runger, 2010, Q2-84]

Solution:

(a) By multiplication rule, there are

$$|\Omega| = 4 \times 3 \times 5 \times 3 \times 5 \tag{4.1}$$

possible designs. The number of designs whose color is red is given by

$$|A| = 1 \times 3 \times 5 \times 3 \times 5.$$

Note that the "4" in (4.1) is replaced by "1" because we only consider one color (red). Therefore,

$$P(A) = \frac{1 \times 3 \times 5 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{1}{4}.}$$

Similarly, $|B| = 4 \times 3 \times 4 \times 3 \times 5$ where the "5" in the middle of (4.1) is replaced by "4" because we can't use the smallest font size. Therefore,

$$P(B) = \frac{4 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{4}{5}.}$$

For the event $A \cap B$, we replace "4" in (4.1) by "1" because we need red color and we replace "5" in the middle of (4.1) by "4" because we can't use the smallest font size. This gives

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{1 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \frac{1 \times 4}{4 \times 5} = \boxed{\frac{1}{5}} = 0.2.$$

Because $P(A \cap B) = P(A)P(B)$, the events A and B are independent.

(b)

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{4}{5} - \frac{1}{5} = \boxed{\frac{17}{20} = 0.85.}$

(ii) **Method 1**: $P(A \cup B^c) = 1 - P\left((A \cup B^c)^c\right) = 1 - P(A^c \cap B)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$. Hence, $P(A^c \cup B^c) = 1 - P(A^c)P(B) = 1 - \frac{3}{4}\frac{4}{5} = \frac{2}{5} = \boxed{0.4.}$

Method 2: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = B^c \cup (A \cap B)$. Therefore,

$$P(A \cup B^c) = P(B^c) + P(A \cap B) = 1 - P(B) + P(A)P(B) = 1 - \frac{4}{5} + \frac{1}{4}\frac{4}{5} = \frac{2}{5}.$$

Method 3: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = A \cup (A^c \cap B^c)$. Therefore, $P(A \cup B^c) = P(A) + P(A^c \cap B^c)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B^c$. Hence,

$$P(A \cup B^c) = P(A) + P(A^c)P(B^c) = P(A) + (1 - P(A))(1 - P(B)) = \frac{1}{4} + \frac{3}{4}\frac{1}{5} = \frac{2}{5}.$$

(iii) **Method 1**: $P(A^c \cup B^c) = 1 - P\left((A^c \cup B^c)^c\right) = 1 - P(A \cap B) = 1 - 0.2 = \boxed{0.8.}$
Method 2: From the Venn diagram, note that $A^c \cup B^c$ can be expressed as a disjoint union: $A^c \cup B^c = (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c)$. Therefore,

$$P\left(A^c \cup B^c\right) = P\left(A^c \cap B\right) + P\left(A \cap B^c\right) + P\left(A^c \cap B^c\right).$$

Now, because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$, $A \perp\!\!\!\perp B^c$, and $A^c \perp\!\!\!\perp B^c$. Hence,

$$\begin{aligned} P\left(A^c \cup B^c\right) &= P\left(A^c\right) P\left(B\right) + P\left(A\right) P\left(B^c\right) + P\left(A^c\right) P\left(B^c\right) \\ &= \left(1 - P\left(A\right)\right) P\left(B\right) + P\left(A\right)\left(1 - P\left(B\right)\right) + \left(1 - P\left(A\right)\right)\left(1 - P\left(B\right)\right) \\ &= \frac{3}{4} \times \frac{4}{5} + \frac{1}{4} \times \frac{1}{5} + \frac{3}{4} \times \frac{1}{5} = \frac{16}{20} = \frac{4}{5} \end{aligned}$$

Problem 2. In this question, each experiment has equiprobable outcomes.

(a) Let $\Omega = \{1, 2, 3, 4\}$, $A_1 = \{1, 2\}$, $A_2 = \{1, 3\}$, $A_3 = \{2, 3\}$.

(i) Determine whether $P\left(A_i \cap A_j\right) = P\left(A_i\right) P\left(A_j\right)$ for all $i \neq j$.

(ii) Check whether $P\left(A_1 \cap A_2 \cap A_3\right) = P\left(A_1\right) P\left(A_2\right) P\left(A_3\right)$.

(iii) Are A_1, A_2, and A_3 independent?

(b) Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2, 3, 4\}$, $A_2 = A_3 = \{4, 5, 6\}$.

(i) Check whether $P\left(A_1 \cap A_2 \cap A_3\right) = P\left(A_1\right) P\left(A_2\right) P\left(A_3\right)$.

(ii) Check whether $P\left(A_i \cap A_j\right) = P\left(A_i\right) P\left(A_j\right)$ for all $i \neq j$.

(iii) Are A_1, A_2, and A_3 independent?

Solution:

(a) We have $P(A_i) = \frac{1}{2}$ and $P(A_i \cap A_j) = \frac{1}{4}$.

(i) $P(A_i \cap A_j) = P(A_i)P(A_j)$ for any $i \neq j$.

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$. Hence, $P\left(A_1 \cap A_2 \cap A_3\right) = 0$, which is *not* the same as $P\left(A_1\right) P\left(A_2\right) P\left(A_3\right)$.

(iii) No.

Remark: This counter-example shows that pairwise independence does not imply independence.

(b) We have $P(A_1) = \frac{4}{6} = \frac{2}{3}$ and $P(A_2) = P(A_3) = \frac{3}{6} = \frac{1}{2}$.

(i) $A_1 \cap A_2 \cap A_3 = \{4\}$. Hence, $P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}$.
$P(A_1)\,P(A_2)\,P(A_3) = \frac{2}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{6}$.
Hence, $P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$.

(ii) $P(A_2 \cap A_3) = P(A_2) = \frac{1}{2} \neq P(A_2)P(A_3)$
$P(A_1 \cap A_2) = p(4) = \frac{1}{6} \neq P(A_1)P(A_2)$
$P(A_1 \cap A_3) = p(4) = \frac{1}{6} \neq P(A_1)P(A_3)$
Hence, $P(A_i \cap A_j) \neq P(A_i)\,P(A_j)$ for all $i \neq j$.

(iii) No.

Remark: This counter-example shows that one product condition does not imply independence.

Problem 3. In an experiment, A, B, C, and D are events with probabilities $P(A \cup B) = \frac{5}{8}$, $P(A) = \frac{3}{8}$, $P(C \cap D) = \frac{1}{3}$, and $P(C) = \frac{1}{2}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find

(i) $P(A \cap B)$
(ii) $P(B)$
(iii) $P(A \cap B^c)$
(iv) $P(A \cup B^c)$

(b) Are A and B independent?

(c) Find

(i) $P(D)$
(ii) $P(C \cap D^c)$
(iii) $P(C^c \cap D^c)$
(iv) $P(C|D)$
(v) $P(C \cup D)$
(vi) $P(C \cup D^c)$

(d) Are C and D^c independent?

Solution:

(a)

(i) Because $A \perp B$, we have $A \cap B = \emptyset$ and hence $P(A \cap B) = \boxed{0}$.

(ii) Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, $P(B) = P(A \cup B) - P(A) + P(A \cap B) = 5/8 - 3/8 + 0 = 2/8 = \boxed{1/4}$.

(iii) $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) = \boxed{3/8}$.

(iv) Start with $P(A \cup B^c) = 1 - P(A^c \cap B)$. Now, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) = 1/4$. Hence, $P(A \cup B^c) = 1 - 1/4 = \boxed{3/4}$.

(b) Events A and B are not independent because $P(A \cap B) \neq P(A)P(B)$.

(c)

(i) Because $C \perp\!\!\!\perp D$, we have $P(C \cap D) = P(C)P(D)$. Hence, $P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1/3}{1/2} = \boxed{2/3}$.

(ii) **Method 1**: $P(C \cap D^c) = P(C) - P(C \cap D) = 1/2 - 1/3 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C \perp\!\!\!\perp D^c$. Hence, $P(C \cap D^c) = P(C)P(D^c) = \frac{1}{2}\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iii) **Method 1**: First, we find $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = 5/6$. Hence, $P(C^c \cap D^c) = 1 - P(C \cup D) = 1 - 5/6 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C^c \perp\!\!\!\perp D^c$. Hence, $P(C^c \cap D^c) = P(C^c)P(D^c) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iv) Because $C \perp\!\!\!\perp D$, we have $P(C|D) = P(C) = \boxed{1/2}$.

(v) In part (iii), we already found $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = \boxed{5/6}$.

(vi) **Method 1**: $P(C \cup D^c) = 1 - P(C^c \cap D) = 1 - P(C^c)P(D) = 1 - \frac{1}{2}\frac{2}{3} = \boxed{2/3}$. Note that we use the fact that $C^c \perp\!\!\!\perp D$ to get the second equality.
Method 2: Alternatively, $P(C \cup D^c) = P(C) + P(D^c) - P(C \cap D^C)$. From (i), we have $P(D) = 2/3$. Hence, $P(D^c) = 1 - 2/3 = 1/3$. From (ii), we have $P(C \cap D^C) = 1/6$. Therefore, $P(C \cup D^c) = 1/2 + 1/3 - 1/6 = 2/3$.

(d) Yes. We know that if $C \perp\!\!\!\perp D$, then $C \perp\!\!\!\perp D^c$.

Figure 4.1: Circuit for Problem 4

Problem 4. Series Circuit: The circuit in Figure 4.1 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-32]

Solution: Let L and R denote the events that the left and right devices operate, respectively. For a path to exist, both need to operate. Therefore, the probability that the circuit operates is $P(L \cap R)$.

We are told that $L^c \perp\!\!\!\perp R^c$. This is equivalent to $L \perp\!\!\!\perp R$. By their independence,

$$P(L \cap R) = P(L)P(R) = 0.8 \times 0.9 = \boxed{0.72}.$$

Extra Questions

Here are optional questions for those who want more practice.

Problem 5. Show that if A and B are independent events, then so are A and B^c, A^c and B, and A^c and B^c.

Solution: To show that two events C_1 and C_2 are independent, we need to show that $P(C_1 \cap C_2) = P(C_1)P(C_2)$.

(a) Note that

$$P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B).$$

Because $A \perp\!\!\!\perp B$, the last term can be factored in to $P(A)P(B)$ and hence

$$P(A \cap B^c) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$$

(b) By interchanging the role of A and B in the previous part, we have

$$P(A^c \cap B) = P(B \cap A^c) = P(B)\,P(A^c).$$

(c) From set theory, we know that $A^c \cap B^c = (A \cup B)^c$. Therefore,

$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B),$$

where, for the last equality, we use

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

which is discussed in class.

Because $A \perp\!\!\!\perp B$, we have

$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c). \end{aligned}$$

Remark: By interchanging the roles of A and A^c and/or B and B^c, it follows that if any one of the four pairs is independent, then so are the other three. [Gubner, 2006, p.31]

Problem 6. Anne and Betty go fishing. Find the conditional probability that Anne catches no fish given that at least one of them catches no fish. Assume they catch fish independently and that each has probability $0 < p < 1$ of catching no fish. [Gubner, 2006, Q2.62]

Hint: Let A be the event that Anne catches no fish and B be the event that Betty catches no fish. Observe that the question asks you to evaluate $P(A|(A \cup B))$.

Solution: From the question, we know that A and B are independent. The event "at least one of the two women catches nothing" can be represented by $A \cup B$. So we have

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B) - P(A)P(B)} = \frac{p}{2p - p^2} = \boxed{\frac{1}{2 - p}}.$$

Problem 7. The circuit in Figure 4.2 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-34]

Solution: Let T and B denote the events that the top and bottom devices operate, respectively. There is a path if at least one device operates. Therefore, the probability that the circuit operates is $P(T \cup B)$. Note that

$$P(T \cup B) = 1 - P((T \cup B)^c) = 1 - P(T^c \cap B^c).$$

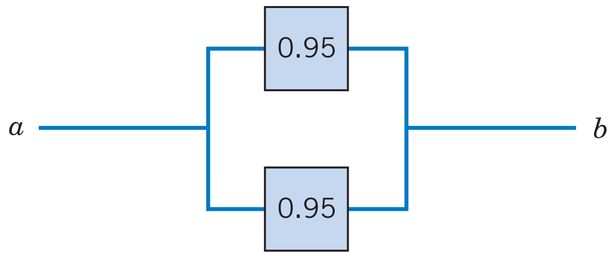

Figure 4.2: Circuit for Problem 7

We are told that $T^c \perp\!\!\!\perp B^c$. By their independence,

$$P(T^c \cap B^c) = P(T^c)P(B^c) = (1-0.95)\times(1-0.95) = 0.05^2 = 0.0025.$$

Therefore,

$$P(T \cup B) = 1 - P(T^c \cap B^c) = 1 - 0.0025 = \boxed{0.9975}.$$

Problem 8. The circuit in Figure 4.3 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-35]

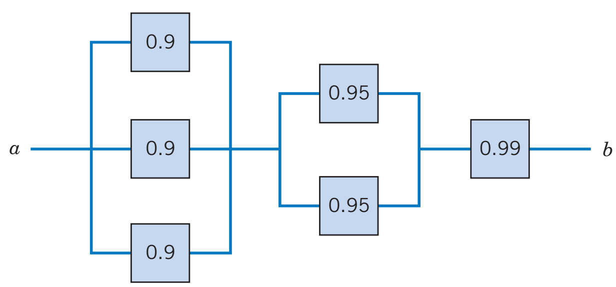

Figure 4.3: Circuit for Problem 8

Solution: The solution can be obtained from a partition of the graph into three columns. Let L denote the event that there is a path of functional devices only through the three units on the left. From the independence and based upon Problem 7,

$$P(L) = 1 - (1-0.9)^3 = 1 - 0.1^3 = 0.999.$$

Similarly, let M denote the event that there is a path of functional devices only through the two units in the middle. Then,

$$P(M) = 1 - (1 - 0.95)^2 = 1 - 0.05^2 = 1 - 0.0025 = 0.9975.$$

Finally, the probability that there is a path of functional devices only through the one unit on the right is simply the probability that the device functions, namely, 0.99.

Therefore, with the independence assumption used again, along with similar reasoning to the solution of Problem 4, the solution is

$$0.999 \times 0.9975 \times 0.99 = 0.986537475 \approx \boxed{0.987}.$$

Problem 9. Show that

(a) $P(A \cap B \cap C) = P(A) \times P(B|A) \times P(C|A \cap B)$.

(b) $P(B \cap C|A) = P(B|A)P(C|B \cap A)$

Solution:

(a) We can see directly from the definition of $P(B|A)$ that

$$P(A \cap B) = P(A)P(B|A).$$

Similarly, when we consider event $A \cap B$ and event C, we have

$$P(A \cap B \cap C) = P(A \cap B)\, P(C|A \cap B).$$

Combining the two equalities above, we have

$$P(A \cap B \cap C) = P(A) \times P(B|A) \times P(C|A \cap B).$$

(b) By definition,

$$P(B \cap C|A) = \frac{P(A \cap B \cap C)}{P(A)}.$$

Substitute $P(A \cap B \cap C)$ from part (a) to get

$$P(B \cap C|A) = \frac{P(A) \times P(B|A) \times P(C|A \cap B)}{P(A)} = P(B|A) \times P(C|A \cap B).$$