ECS 315: Probability and Random Processes **2014/1**

HW Solution 2 — Due: Sep 4

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Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. If A, B, and C are disjoint events with $P(A) = 0.2$, $P(B) = 0.3$ and $P(C) = 0.4$, determine the following probabilities:

(a) $P(A \cup B \cup C)$

(b) $P(A \cap B \cap C)$

(c) $P(A \cap B)$

(d) $P((A \cup B) \cap C)$

(e) $P(A^c \cap B^c \cap C^c)$

[Montgomery and Runger, 2010, Q2-75]

Solution:

(a) Because A, B, and C are disjoint, $P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.3 + 0.2 + 0.4 = \boxed{0.9.}$

(b) Because A, B, and C are disjoint, $A \cap B \cap C = \emptyset$ and hence $P(A \cap B \cap C) = P(\emptyset) = \boxed{0}$.

(c) Because A and B are disjoint, $A \cap B = \emptyset$ and hence $P(A \cap B) = P(\emptyset) = \boxed{0}$.

(d) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. By the disjointness among A, B, and C, we have $(A \cap C) \cup (B \cap C) = \emptyset \cup \emptyset = \emptyset$. Therefore, $P((A \cup B) \cap C) = P(\emptyset) = \boxed{0}$.

(e) From $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$, we have $P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C) = 1 - 0.9 = \boxed{0.1.}$

Problem 2. The sample space of a random experiment is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

[Montgomery and Runger, 2010, Q2-55]

Solution:

(a) Recall that the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Therefore,

$$\begin{aligned} P(A) &= P(\{a, b, c\}) = P(\{a\}) + P(\{b\}) + P(\{c\}) \\ &= 0.1 + 0.1 + 0.2 = \boxed{0.4.} \end{aligned}$$

(b) Again, the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Thus,

$$\begin{aligned} P(B) &= P(\{c, d, e\}) = P(\{c\}) + P(\{d\}) + P(\{e\}) \\ &= 0.2 + 0.4 + 0.2 = \boxed{0.8.} \end{aligned}$$

(c) Applying the complement rule, we have $P(A^c) = 1 - P(A) = 1 - 0.4 = \boxed{0.6.}$

(d) Note that $A \cup B = \Omega$. Hence, $P(A \cup B) = P(\Omega) = \boxed{1.}$

(e) $P(A \cap B) = P(\{c\}) = \boxed{0.2.}$

Problem 3.

(a) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Find the range of possible values for $P(A \cap B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.21]

(b) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Find the range of possible values for $P(A \cup B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.22]

Solution:

(a) We will try to derive general bounds for $P(A \cap B)$.

First, recall[1], from the lecture notes, that "$P(A \cap B)$ can not exceed $P(A)$ and $P(B)$":

$$P(A \cap B) \leq \min\{P(A), P(B)\}. \tag{2.1}$$

On the other hand, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{2.2}$$

Now, $P(A \cup B)$ is a probability and hence its value must be between 0 and 1:

$$0 \leq P(A \cup B) \leq 1 \tag{2.3}$$

Combining (2.3) with (2.2) gives

$$P(A) + P(B) - 1 \leq P(A \cap B) \leq P(A) + P(B). \tag{2.4}$$

The second inequality in (2.4) is not useful because (2.1) gives a better[2] bound. So, we will replace the second inequality with (2.1):

$$P(A) + P(B) - 1 \leq P(A \cap B) \leq \min\{P(A), P(B)\}. \tag{2.5}$$

Finally, $P(A \cap B)$ is also a probability and hence it must be between 0 and 1:

$$0 \leq P(A \cap B) \leq 1 \tag{2.6}$$

Combining (2.6) and (2.5), we have

$$\max\{(P(A) + P(B) - 1), 0\} \leq P(A \cap B) \leq \min\{P(A), P(B), 1\}.$$

[1] Again, to see this, note that $A \cap B \subset A$ and $A \cap B \subset B$. Hence, we know that $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$.

[2] When we already know that a number is less than 3, learning that it is less than 5 does not give us any new information.

Note that number one at the end of the expression above is not necessary because the two probabilities under minimization can not exceed 1 themselves. In conclusion,

$$\max\{(P(A)+P(B)-1),0\} \le P(A\cap B) \le \min\{P(A),P(B)\}.$$

Plugging in the value $P(A)=\frac{1}{2}$ and $P(B)=\frac{2}{3}$ gives the range $\boxed{\left[\frac{1}{6},\frac{1}{2}\right]}$.

Note that the upper-bound can be obtained by constructing an example which has $A\subset B$. The lower-bound can be obtained by considering an example where $A\cup B=\Omega$.

(b) We will try to derive general bounds for $P(A\cup B)$.

By monotonicity, because both A and B are subset of $A\cup B$, we must have

$$P(A\cup B) \ge \max\{P(A),P(B)\}.$$

On the other hand, we know, from the finite sub-additivity property, that

$$P(A\cup B) \le P(A)+P(B).$$

Therefore,

$$\max\{P(A),P(B)\} \le P(A\cup B) \le P(A)+P(B).$$

Being a probability, $P(A\cup B)$ must be between 0 and 1. Hence,

$$\max\{P(A),P(B),0\} \le P(A\cup B) \le \min\left\{\left(P(A)+P(B)\right),1\right\}.$$

Note that number 0 is not needed in the aximization because the two probabilities involved are automatically ≥ 0 themselves.

In conclusion,

$$\max\{P(A),P(B)\} \le P(A\cup B) \le \min\{(P(A)+P(B)),1\}.$$

Plugging in the value $P(A)=\frac{1}{2}$ and $P(B)=\frac{1}{3}$, we have

$$P(A\cup B) \in \boxed{\left[\frac{1}{2},\frac{5}{6}\right]}.$$

The upper-bound can be obtained by making $A\perp B$. The lower-bound is achieved when $B\subset A$.

Problem 4. Let A and B be events for which $P(A)$, $P(B)$, and $P(A \cup B)$ are known. Express the following probabilities in terms of the three known probabilities above.

(a) $P(A \cap B)$

(b) $P(A \cap B^c)$

(c) $P(B \cup (A \cap B^c))$

(d) $P(A^c \cap B^c)$

Solution:

(a) $P(A \cap B) = \boxed{P(A) + P(B) - P(A \cup B)}$. This property is shown in class.

(b) We have seen[3] in class that $P(A \cap B^c) = P(A) - P(A \cap B)$. Plugging in the expression for $P(A \cap B)$ from the previous part, we have

$$P(A \cap B^c) = P(A) - (P(A) + P(B) - P(A \cup B)) = \boxed{P(A \cup B) - P(B)}.$$

Alternatively, we can start from scratch with the set identity $A \cup B = B \cup (A \cap B^c)$ whose union is a disjoint union. Hence,

$$P(A \cup B) = P(B) + P(A \cap B^c).$$

Moving $P(B)$ to the LHS finishes the proof.

(c) $P(B \cup (A \cap B^c)) = \boxed{P(A \cup B)}$ because $A \cup B = B \cup (A \cap B^c)$.

(d) $P(A^c \cap B^c) = \boxed{1 - P(A \cup B)}$ because $A^c \cap B^c = (A \cup B)^c$.

[3]This shows up when we try to derive the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. The key idea is that the set A can be expressed as a disjoint union between $A \cap B$ and $A \cap B^c$. Therefore, by finite additivity, $P(A) = P(A \cap B) + P(A \cap B^c)$. It is easier to visualize this via the Venn diagram.