

HW Solution 12 — Due: Dec 4

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Instructions

- (a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.
- (b) It is important that you try to solve all problems. (5 pt)
The extra questions at the end are optional.
- (c) Late submission will be heavily penalized.
- (d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

	y	2	4	5
x				
1		0.02	0.10	0.08
3		0.08	0.32	0.40

Evaluate the following quantities:

- (a) The marginal pmf $p_X(x)$
- (b) The marginal pmf $p_Y(y)$
- (c) $\mathbb{E}X$
- (d) $\text{Var } X$
- (e) $\mathbb{E}Y$
- (f) $\text{Var } Y$

(g) $P[XY < 6]$

(h) $P[X = Y]$

Solution: The MATLAB codes are provided in the file `P_XY_marginal.m`.

(a) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 0.2, & x = 1 \\ 0.8, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

(b) The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 0.1, & y = 2 \\ 0.42, & y = 4 \\ 0.48, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

(c) $\mathbb{E}X = \sum_x xp_X(x) = 1 \times 0.2 + 3 \times 0.8 = 0.2 + 2.4 = \boxed{2.6}$.

(d) $\mathbb{E}[X^2] = \sum_x x^2 p_X(x) = 1^2 \times 0.2 + 3^2 \times 0.8 = 0.2 + 7.2 = 7.4$. So, $\text{Var } X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = 7.4 - (2.6)^2 = 7.4 - 6.76 = \boxed{0.64}$.

(e) $\mathbb{E}Y = \sum_y yp_Y(y) = 2 \times 0.1 + 4 \times 0.42 + 5 \times 0.48 = 0.2 + 1.68 + 2.4 = \boxed{4.28}$.

(f) $\mathbb{E}[Y^2] = \sum_y y^2 p_Y(y) = 2^2 \times 0.1 + 4^2 \times 0.42 + 5^2 \times 0.48 = 19.12$. So, $\text{Var } Y = \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 = 19.12 - 4.28^2 = \boxed{0.8016}$.

(g) Among the 6 possible pairs of (x, y) shown in the joint pmf matrix, only the pairs $(1, 2)$, $(1, 4)$, $(1, 5)$ satisfy $xy < 6$. Therefore, $[XY < 6] = [X = 1]$ which implies $P[XY < 6] = P[X = 1] = \boxed{0.2}$.

(h) Among the 6 possible pairs of (x, y) shown in the joint pmf matrix, there is no pair which has $x = y$. Therefore, $P[X = Y] = \boxed{0}$.

Problem 2. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

	y	2	4	5
x				
1	[0.02	0.10	0.08]
3	[0.08	0.32	0.40]

(a) Evaluate the following quantities:

- (i) $\mathbb{E}[XY]$
- (ii) $\mathbb{E}[(X - 3)(Y - 2)]$
- (iii) $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)]$
- (iv) $\text{Cov}[X, Y]$
- (v) $\rho_{X,Y}$

Hint: Write down the formulas then use MATLAB or Excel to compute them.

(b) Find $\rho_{X,X}$

(c) Calculate the following quantities using the values of $\text{Var } X$, $\text{Cov}[X, Y]$, and $\rho_{X,Y}$ that you got earlier.

- (i) $\text{Cov}[3X + 4, 6Y - 7]$
- (ii) $\rho_{3X+4, 6Y-7}$
- (iii) $\text{Cov}[X, 6X - 7]$
- (iv) $\rho_{X, 6X-7}$

Solution: The MATLAB codes are provided in the file P_XY_EVarCov.m.

(a)

- (i) From MATLAB, $\mathbb{E}[XY] = \boxed{11.16}$.
- (ii) From MATLAB, $\mathbb{E}[(X - 3)(Y - 2)] = \boxed{-0.88}$.
- (iii) From MATLAB, $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)] = \boxed{104}$.
- (iv) From MATLAB, $\text{Cov}[X, Y] = \boxed{0.032}$.

- (v) From MATLAB, $\rho_{X,Y} = \boxed{0.044677}$.
- (b) $\rho_{X,X} = \frac{\text{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\text{Var}[X]}{\sigma_X^2} = \boxed{1}$.
- (c)
- (i) $\text{Cov}[3X + 4, 6Y - 7] = 3 \times 6 \times \text{Cov}[X, Y] \approx 3 \times 6 \times 0.032 \approx \boxed{0.576}$.
- (ii) Note that

$$\begin{aligned} \rho_{aX+b, cY+d} &= \frac{\text{Cov}[aX + b, cY + d]}{\sigma_{aX+b} \sigma_{cY+d}} \\ &= \frac{ac \text{Cov}[X, Y]}{|a| \sigma_X |c| \sigma_Y} = \frac{ac}{|ac|} \rho_{X,Y} = \text{sign}(ac) \times \rho_{X,Y}. \end{aligned}$$

Hence, $\rho_{3X+4, 6Y-7} = \text{sign}(3 \times 6) \rho_{X,Y} = \rho_{X,Y} = \boxed{0.0447}$.

- (iii) $\text{Cov}[X, 6X - 7] = 1 \times 6 \times \text{Cov}[X, X] = 6 \times \text{Var}[X] \approx \boxed{3.84}$.
- (iv) $\rho_{X, 6X-7} = \text{sign}(1 \times 6) \times \rho_{X,X} = \boxed{1}$.

Problem 3. Suppose $X \sim \text{binomial}(5, 1/3)$, $Y \sim \text{binomial}(7, 4/5)$, and $X \perp\!\!\!\perp Y$. Evaluate the following quantities.

- (a) $\mathbb{E}[(X - 3)(Y - 2)]$
- (b) $\text{Cov}[X, Y]$
- (c) $\rho_{X,Y}$

Solution:

- (a) First, because X and Y are independent, we have $\mathbb{E}[(X - 3)(Y - 2)] = \mathbb{E}[X - 3] \mathbb{E}[Y - 2]$. Recall that $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$. Therefore, $\mathbb{E}[X - 3] \mathbb{E}[Y - 2] = (\mathbb{E}[X] - 3)(\mathbb{E}[Y] - 2)$. Now, for Binomial(n, p), the expected value is np . So,

$$(\mathbb{E}[X] - 3)(\mathbb{E}[Y] - 2) = \left(5 \times \frac{1}{3} - 3\right) \left(7 \times \frac{4}{5} - 2\right) = -\frac{4}{3} \times \frac{18}{5} = \boxed{-\frac{24}{5}} = -4.8.$$

- (b) $\text{Cov}[X, Y] = \boxed{0}$ because $X \perp\!\!\!\perp Y$.
- (c) $\rho_{X,Y} = \boxed{0}$ because $\text{Cov}[X, Y] = 0$

Problem 4. Suppose we know that $\sigma_X = \frac{\sqrt{21}}{10}$, $\sigma_Y = \frac{4\sqrt{6}}{5}$, $\rho_{X,Y} = -\frac{1}{\sqrt{126}}$.

- (a) Find $\text{Var}[X + Y]$.
 (b) Find $\mathbb{E}[(Y - 3X + 5)^2]$. Assume $\mathbb{E}[Y - 3X + 5] = 1$.

Solution:

- (a) First, we know that $\text{Var } X = \sigma_X^2 = \frac{21}{100}$, $\text{Var } Y = \sigma_Y^2 = \frac{96}{25}$, and $\text{Cov}[X, Y] = \rho_{X,Y} \times \sigma_X \times \sigma_Y = -\frac{2}{25}$. Now,

$$\begin{aligned} \text{Var}[X + Y] &= \mathbb{E}[\left((X + Y) - \mathbb{E}[X + Y]\right)^2] = \mathbb{E}[\left((X - \mathbb{E}X) + (Y - \mathbb{E}Y)\right)^2] \\ &= \mathbb{E}[(X - \mathbb{E}X)^2] + 2\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] + \mathbb{E}[(Y - \mathbb{E}Y)^2] \\ &= \text{Var } X + 2\text{Cov}[X, Y] + \text{Var } Y \\ &= \boxed{\frac{389}{100}} = 3.89. \end{aligned}$$

Remark: It is useful to remember that

$$\text{Var}[X + Y] = \text{Var } X + 2\text{Cov}[X, Y] + \text{Var } Y.$$

Note that when X and Y are uncorrelated, $\text{Var}[X + Y] = \text{Var } X + \text{Var } Y$. This simpler formula also holds when X and Y are independence because independence is a stronger condition.

- (b) First, we write

$$Y - aX - b = (Y - \mathbb{E}Y) - a(X - \mathbb{E}X) - \underbrace{(a\mathbb{E}X + b - \mathbb{E}Y)}_c.$$

Now, using the expansion

$$(u + v + t)^2 = u^2 + v^2 + t^2 + 2uv + 2ut + 2vt,$$

we have

$$\begin{aligned} (Y - aX - b)^2 &= (Y - \mathbb{E}Y)^2 + a^2(X - \mathbb{E}X)^2 + c^2 \\ &\quad - 2a(X - \mathbb{E}X)(Y - \mathbb{E}Y) - 2c(Y - \mathbb{E}Y) + 2a(X - \mathbb{E}X)c. \end{aligned}$$

Recall that $\mathbb{E}[X - \mathbb{E}X] = \mathbb{E}[Y - \mathbb{E}Y] = 0$. Therefore,

$$\mathbb{E}[(Y - aX - b)^2] = \text{Var } Y + a^2 \text{Var } X + c^2 - 2a\text{Cov}[X, Y]$$

Plugging back the value of c , we have

$$\mathbb{E} [(Y - aX - b)^2] = \text{Var } Y + a^2 \text{Var } X + (\mathbb{E} [(Y - aX - b)])^2 - 2a \text{Cov} [X, Y].$$

Here, $a = 3$ and $b = -5$. Plugging these values along with the given quantities into the formula gives

$$\mathbb{E} [(Y - aX - b)^2] = \boxed{\frac{721}{100}} = 7.21.$$

Extra Question

Here are optional questions for those who want extra practice.

Problem 5. The input X and output Y of a system subject to random perturbations are described probabilistically by the joint pmf $p_{X,Y}(x, y)$, where $x = 1, 2, 3$ and $y = 1, 2, 3, 4, 5$. Let P denote the joint pmf matrix whose i, j entry is $p_{X,Y}(i, j)$, and suppose that

$$P = \frac{1}{71} \begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

- Find the marginal pmfs $p_X(x)$ and $p_Y(y)$.
- Find $\mathbb{E}X$
- Find $\mathbb{E}Y$
- Find $\text{Var } X$
- Find $\text{Var } Y$

Solution: All of the calculations in this question are simply plugging numbers into appropriate formula. The MATLAB codes are provided in the file `P_XY_marginal_2.m`.

- The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 26/71, & x = 1 \\ 25/71, & x = 2 \\ 20/71, & x = 3 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.3662, & x = 1 \\ 0.3521, & x = 2 \\ 0.2817, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 13/71, & y = 1 \\ 8/71, & y = 2 \\ 21/71, & y = 3 \\ 15/71, & y = 4 \\ 14/71, & y = 5 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.1831, & y = 1 \\ 0.1127, & y = 2 \\ 0.2958, & y = 3 \\ 0.2113, & y = 4 \\ 0.1972, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

(b) $\mathbb{E}X = \frac{136}{71} \approx 1.9155$

(c) $\mathbb{E}Y = \frac{222}{71} \approx 3.1268$

(d) $\text{Var } X = \frac{3230}{5041} \approx 0.6407$

(e) $\text{Var } Y = \frac{9220}{5041} \approx 1.8290$

Problem 6. A webpage server can handle r requests per day. Find the probability that the server gets more than r requests at least once in n days. Assume that the number of requests on day i is $X_i \sim \mathcal{P}(\alpha)$ and that X_1, \dots, X_n are independent.

Solution: [Gubner, 2006, Ex 2.10]

$$\begin{aligned} P \left[\bigcup_{i=1}^n [X_i > r] \right] &= 1 - P \left[\bigcap_{i=1}^n [X_i \leq r] \right] = 1 - \prod_{i=1}^n P[X_i \leq r] \\ &= 1 - \prod_{i=1}^n \left(\sum_{k=0}^r \frac{\alpha^k e^{-\alpha}}{k!} \right) = \boxed{1 - \left(\sum_{k=0}^r \frac{\alpha^k e^{-\alpha}}{k!} \right)^n}. \end{aligned}$$

Problem 7. Suppose $X \sim \text{binomial}(5, 1/3)$, $Y \sim \text{binomial}(7, 4/5)$, and $X \perp\!\!\!\perp Y$.

(a) A vector describing the pmf of X can be created by the MATLAB expression:

$$\mathbf{x} = 0:5; \mathbf{pX} = \text{binopdf}(\mathbf{x}, 5, 1/3).$$

What is the expression that would give \mathbf{pY} , a corresponding vector describing the pmf of Y ?

(b) Use \mathbf{pX} and \mathbf{pY} from part (a), how can you create the joint pmf matrix in MATLAB? Do not use “for-loop”, “while-loop”, “if statement”. Hint: Multiply them in an appropriate orientation.

(c) Use MATLAB to evaluate the following quantities. Again, do not use “for-loop”, “while-loop”, “if statement”.

- (i) $\mathbb{E}X$
- (ii) $P[X = Y]$
- (iii) $P[XY < 6]$

Solution: The MATLAB codes are provided in the file P_XY_jointfromMarginal_indp.m.

(a) `y = 0:7; pY = binopdf(y,7,4/5);`

(b) `P = pX.'*pY;`

(c)

(i) $\mathbb{E}X = 1.667$

(ii) $P[X = Y] = 0.0121$

(iii) $P[XY < 6] = 0.2727$

Problem 8. Suppose $\text{Var } X = 5$. Find $\text{Cov}[X, X]$ and $\rho_{X,X}$.

Solution:

(a) $\text{Cov}[X, X] = \mathbb{E}[(X - \mathbb{E}X)(X - \mathbb{E}X)] = \mathbb{E}[(X - \mathbb{E}X)^2] = \text{Var } X = 5$.

(b) $\rho_{X,X} = \frac{\text{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\text{Var } X}{\sigma_X^2} = \frac{\text{Var } X}{\text{Var } X} = 1$.