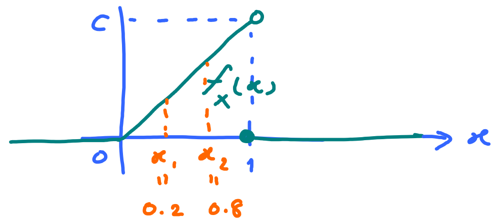
Q1 Meaning of pdf

Wednesday, November 12, 2014 3:21 PM

(a)

$$f_X(x) = \begin{cases} cx, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$



$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \frac{1}{2} \times 1 \times c = 1 \Rightarrow c = 2$$

(b) $f_X(x_1) = f_X(0.2) = c \times 0.2 = 0.4$

$f_X(x_2) = f_X(0.8) = c \times 0.8 = 1.6 \leftarrow$ Remark: f_X does not have to be between 0 and 1 because it is not probability.

(c) $P[0.19 < X < 0.21] = \int_{0.19}^{0.21} f_X(x)\,dx = \cancel{c}\frac{x^2}{\cancel{2}}\Big|_{0.19}^{0.21} = 0.21^2 - 0.19^2 = 0.008$

$P[0.79 < X < 0.81] = \int_{0.79}^{0.81} f_X(x)\,dx = \cancel{c}\frac{x^2}{\cancel{2}}\Big|_{0.79}^{0.81} = 0.81^2 - 0.79^2 = 0.032$

In the approximation, we can plug-in any $x \in (0.19, 0.21)$. Here, we plug-in the middle point.

(d) $P[0.19 < X < 0.21] \approx f_X(0.2)\underbrace{\Delta x}_{0.21-0.19=0.02} = 2 \times (0.2) \times 0.02 = 0.008$

$P[0.79 < X < 0.81] \approx f_X(0.8)\underbrace{\Delta x}_{0.81-0.79=0.02} = 2 \times 0.8 \times 0.02 = 0.032$

They are exactly the same as the answers in part (c).

Q2 Meaning of pdf

Wednesday, November 12, 2014 3:21 PM

(a) $f_X(x) = \begin{cases} cx^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \underbrace{\int_0^1 cx^2\,dx}_{= c\frac{x^3}{3}\Big|_0^1 = \frac{c}{3}} = 1 \Rightarrow c = 3$$

(b) $f_X(x_1) = f_X(0.2) = c \times 0.2^2 = 3 \times 0.04 = 0.12$

$f_X(x_2) = f_X(0.8) = c \times 0.8^2 = 3 \times 0.64 = 1.92$

(c) $P[0.19 < X < 0.21] = \int_{0.19}^{0.21} f_X(x)\,dx = \cancel{3}\frac{x^3}{\cancel{3}}\Big|_{0.19}^{0.21} = 0.21^3 - 0.19^3 = 0.002402$

$P[0.79 < X < 0.81] = \int_{0.79}^{0.81} f_X(x)\,dx = \cancel{3}\frac{x^3}{\cancel{3}}\Big|_{0.79}^{0.81} = 0.81^3 - 0.79^3 = 0.038402$

(d) $P[0.19 < X < 0.21] \approx f_X(0.2)\underbrace{\Delta x}_{0.21-0.19=0.02} = 0.12 \times 0.02 = 0.0024$

from (b)

$P[0.79 < X < 0.81] \approx f_X(0.8)\underbrace{\Delta x}_{0.81-0.79=0.02} = 1.92 \times 0.02 = 0.0384$

They are almost the same as the answers in part (c). (wow!)

Q3 Meaning of pdf

Thursday, November 20, 2014 8:27 PM

Recall that $P[a < X < b] \approx f_X(x)\underbrace{\Delta x}_{b-a}$

"any" x in the interval (a,b)

The approximation will be accurate when Δx is small and $f_X(x)$ is "nice" enough.

We pick "1" and "2" here because at the end, we want $f_X(1)/f_X(2)$.

Therefore, $P[0.99 < X < 1.03] \approx f_X(1)(1.03-0.99) = f_X(1)\times 0.04$

$P[1.98 < X < 2.01] \approx f_X(2)(2.01-1.98) = f_X(2)\times 0.03$

Hence,
$$\frac{P[0.99 < X < 1.03]}{P[1.98 < X < 2.01]} \approx \frac{f_X(1)\times 0.04}{f_X(2)\times 0.03} = \frac{f_X(1)}{f_X(2)}\times\frac{3}{4}$$

We are given that this ratio is $\frac{0.1}{0.05} = 2$

$$\Rightarrow \frac{f_X(2)}{f_X(1)} = \frac{3}{4}\times\frac{1}{2} = \frac{3}{8}.$$

Q4 SISO

Thursday, November 20, 2014 8:34 PM

Because $X \sim \mathcal{U}(-1,4)$, we have $f_X(x) = \begin{cases} 1/5, & -1 < x < 4, \\ 0, & \text{otherwise.} \end{cases}$

Note that the function $g(x)$ is piece-wise affine. It may be tempting to directly use the formula $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$. However, by having multiple pieces, there can be multiple x values (one for each piece) that satisfy $y = g(x)$ for a given y. Therefore, the formula above can not be applied directly here.

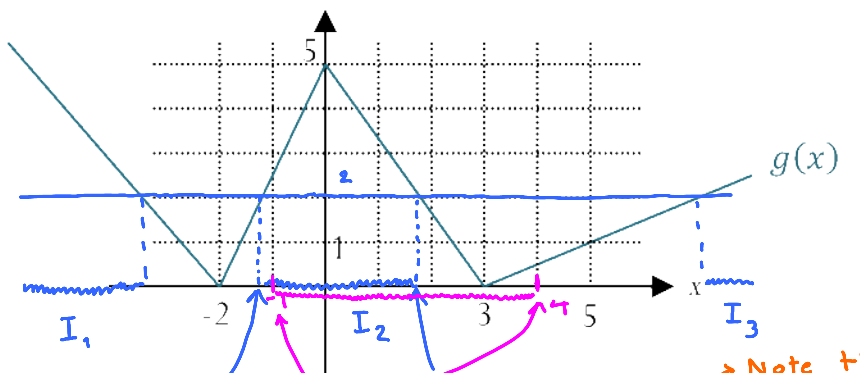
(a) To find $\underbrace{P[Y>2]}_{=P[g(X)>2]}$, we need to find all x such that $g(x) > 2$.

From the plot of $g(x)$, we know that we need $x \in \underbrace{\left(-\frac{6}{5}, \frac{9}{5}\right)}_{I_2}$

or

$x \in I_1$ or $x \in I_3$

We can ignore I_1 and I_3 because they do not overlap with $(-1, 4)$



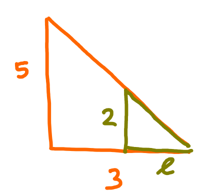
$-2 + \frac{2}{5} \times 2 = -\frac{6}{5}$

$3 - \frac{3}{5} \times 2 = \frac{9}{5}$

Region of nonzero $f_X(x)$

(Note also that $x = -1$ corresponds to $y = 2.5$ in the graph.)

Note that this is easy to see via similar triangles:

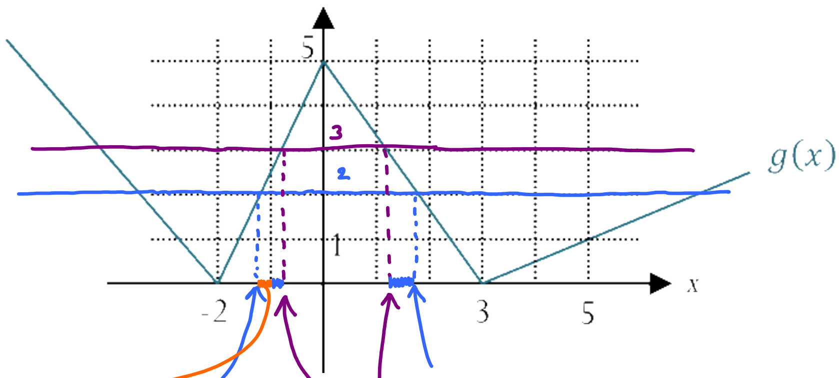


$\frac{5}{3} = \frac{2}{l}$

$$\Rightarrow P[Y>2] = P\left[-\frac{6}{5} < X < \frac{9}{5}\right] = \int_{-6/5}^{9/5} f_X(x)\,dx = \int_{-1}^{9/5} \frac{1}{5}\,dx = \frac{1}{5}\left(\frac{9}{5} - (-1)\right) = \frac{1}{5} \times \frac{14}{5} = \frac{14}{25}.$$

(b) To find $\underbrace{P[2 < Y < 3]}_{= P[2 < g(X) < 3]}$, we need to find all x such that $2 < g(x) < 3$.

From the plot of $g(x)$, we know that we need $x \in \left(-\frac{6}{5}, -\frac{4}{5}\right) \cup \left(\frac{6}{5}, \frac{9}{5}\right)$.



$-2 + \frac{2}{5} \times 2 = -\frac{6}{5}$

$3 - \frac{3}{5} \times 2 = \frac{9}{5}$

$3 - \frac{3}{5} \times 3 = 3 - \frac{9}{5} = \frac{6}{5}$

$-2 + \frac{2}{5} \times 3 = -2 + \frac{6}{5} = -\frac{4}{5}$

This part is outside the region of non zero $f_X(x)$. So, we don't have to consider it.

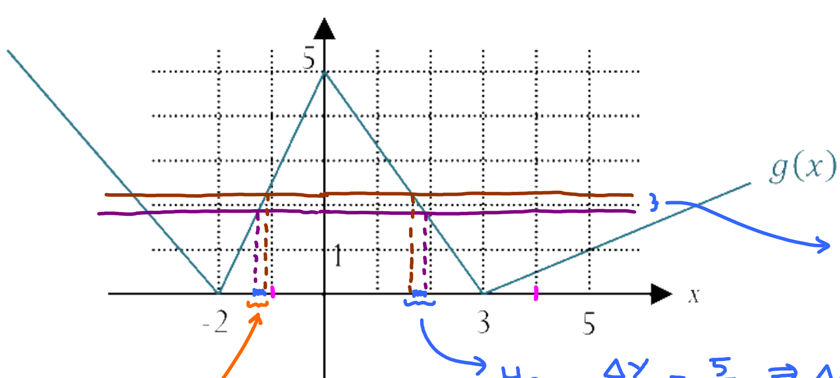
$$\Rightarrow P[2 < Y < 3] = P\left[-\frac{6}{5} < X < -1\right] + P\left[\frac{6}{5} < X < \frac{9}{5}\right]$$

$$= \frac{1}{5}\underbrace{\left(-\frac{4}{5} - (-1)\right)} + \frac{1}{5}\underbrace{\left(\frac{9}{5} - \frac{6}{5}\right)} = \frac{1}{5}\left(\frac{1}{5} + \frac{3}{5}\right) = \frac{4}{25}.$$

Observe that, when X is uniform, we only need the lengths of the intervals here. So, there is no need to actually find the four boundaries.

This observation will simplify our calculation for the next part.

(c) Now, for $P[1.9 < Y < 2.1]$, we use the plot of $g(x)$ to find corresponding interval(s) of x values:



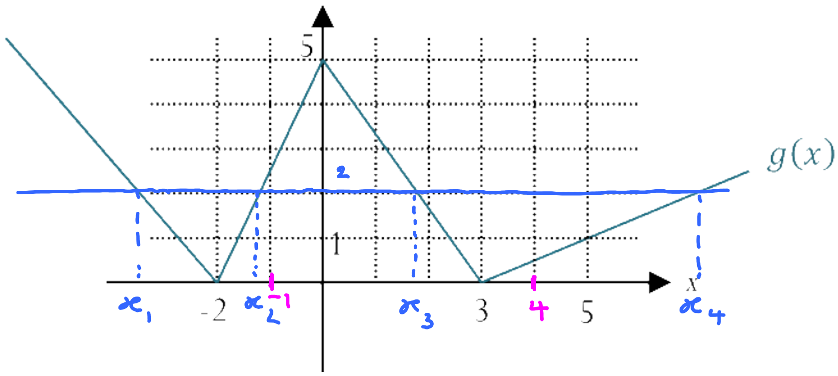
$\Delta y = 2.1 - 1.9 = 0.2$

Here, $\frac{\Delta y}{\Delta x} = \frac{5}{3} \Rightarrow \Delta x = \Delta y \times \frac{3}{5} = 0.2 \times \frac{3}{5} = \frac{0.6}{5}$

We can ignore this region because it is outside the region of non zero $f_X(x)$.

So, $P[1.9 < Y < 2.1] = \frac{1}{5} \times \frac{0.6}{5} = \frac{6}{250} = \frac{3}{125}$

(d) We now apply our shortcut formula:



(i) First, we find x such that $g(x) = 2$.

From the plot above, there is only one x value in $(-1, 4)$ that satisfies $g(x) = 2$ $\hookrightarrow x = x_3$.

Because X is uniform, we don't really need to find the exact value of x_3. Knowing that it is $\in (-1, 4)$, we know that

$$f_X(x_3) = \frac{1}{5}.$$

(ii) Next, we find $g'(x_3)$.

From the plot, we have $g'(x_3) = \frac{5}{3}$.

(iii) Now, we apply the formula:

$$f_Y(y) = \sum_{\substack{x \\ g(x) = y}} \frac{f_X(x)}{|g'(x)|} = \frac{f_X(x_3)}{|g'(x_3)|} = \frac{1/5}{5/3} = \frac{3}{25}.$$

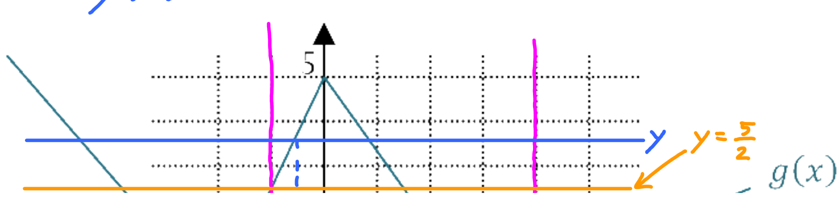
Check:

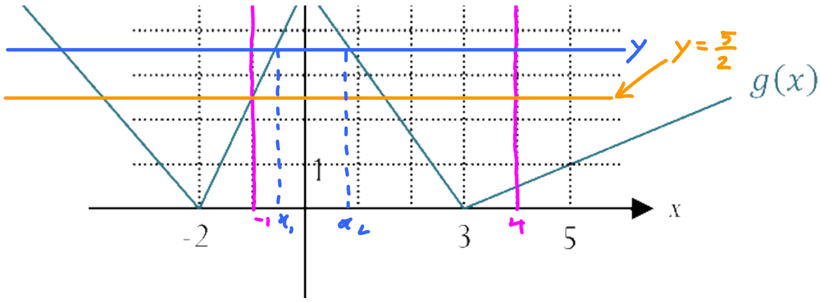
From (c), we know that $P[1.9 < Y < 2.1] = \frac{3}{125}$.

$\approx f_Y(y)\, \Delta y$ — $\Delta y = (2.1 - 1.9) = 0.2$ is small. So, the approximation should be good.

$$\Rightarrow f_Y(y) \approx \frac{3}{125} \times \frac{1}{\Delta y} = \frac{3}{125} \times \frac{1}{0.2} = \frac{3}{125} \times \frac{5}{1} = \frac{3}{25} ✓$$

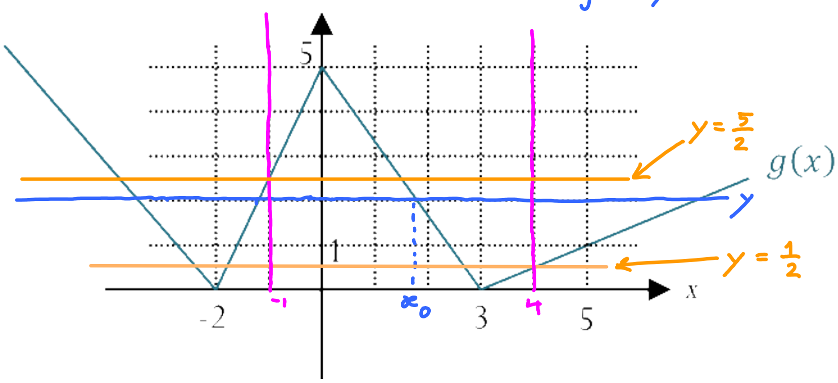
(e) For $y < 0$, or $y > 5$ there is no x that makes $g(x) = y$. So, $f_Y(y) = 0$ for $y < 0$ and $y > 5$.





For $\frac{5}{2} < y < 5$, there are two x values that satisfy $g(x) = y$.
They are shown as x_1 and x_2 in the picture above.
The corresponding slopes are $\frac{5}{2}$ and $-\frac{5}{3}$, respectively.

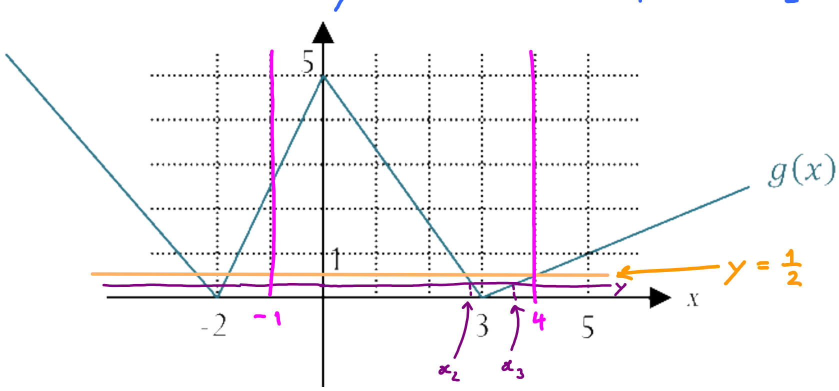
Therefore, $$f_Y(y) = \sum_{\substack{x \\ g(x)=y}} \frac{f_X(x)}{|g'(x)|} = \frac{1/5}{5/2} + \frac{1/5}{5/3} = \frac{5}{25} = \frac{1}{5}.$$



For $\frac{1}{2} \leq y \leq \frac{5}{2}$, there is only one x value that satisfy $g(x) = y$.
It is shown as x_0 in the picture above.
The corresponding slope is $g'(x_0) = -\frac{5}{3}$.

Therefore, $$f_Y(y) = \sum_{\substack{x \\ g(x)=y}} \frac{f_X(x)}{|g'(x)|} = \frac{f_X(x_0)}{|g'(x_0)|} = \frac{1/5}{|-5/3|} = \frac{3}{25}$$

For $0 < y < \frac{1}{2}$, there are two x values that satisfy $g(x) = y$.
They are shown as x_1 and x_2 in the picture below.



The corresponding slopes are
$g'(x_1) = -\frac{5}{3}$ and
$g'(x_2) = \frac{1}{2}$, respectively.

Therefore, $f_Y(y) = \sum \frac{f_X(x)}{} = \frac{1/5}{} + \frac{1/5}{} = \frac{1}{}\left(\frac{3}{5}+2\right) = \frac{13}{}$

Therefore, $f_Y(y) = \sum_{\substack{x \\ g(x)=y}} \frac{f_X(x)}{|g'(x)|} = \frac{\frac{1}{5}}{\frac{5}{3}} + \frac{\frac{1}{5}}{\frac{1}{2}} = \frac{1}{5}\left(\frac{3}{5}+2\right) = \frac{13}{25}$

At $y=5$ and $y=0$, the derivative $g'(x)$ for the solution x of $g(x)=y$ does not exist.

Note that $P[Y=0] = P[X=3] = 0$ and

$P[Y=5] = P[X=0] = 0.$

Therefore, the values of $f_Y(y)$ at $y=0$ and $y=5$ can be set arbitrarily. We choose to set them $=0$.

Conclusion:

$$f_Y(y) = \begin{cases} 13/25, & 0<y<\frac{1}{2}, \\ 3/25, & \frac{1}{2} \le y \le \frac{5}{2}, \\ 1/5, & \frac{5}{2}<y<5, \\ 0, & \text{otherwise.} \end{cases}$$

Check: $\int_{-\infty}^{\infty} f_Y(y)\,dy = \int_0^{1/2} \frac{13}{25}\,dy + \int_{1/2}^{5/2} \frac{3}{25}\,dy + \int_{5/2}^{5} \frac{1}{5}\,dy = \frac{13}{25}\times\frac{1}{2} + \frac{3}{25}\times 2 + \frac{1}{\cancel{5}}\times\frac{\cancel{5}}{2}$

$= \frac{13}{50} + \frac{6}{25} + \frac{1}{2} = 1$ ✓

Check : $P[Y>2] = \int_2^{\infty} f_Y(y)\,dy = \int_2^{5/2} \frac{3}{25}\,dy + \int_{5/2}^{5} \frac{1}{5}\,dy = \frac{3}{25}\times\frac{1}{2} + \frac{1}{\cancel{5}}\times\frac{\cancel{5}}{2}$

$= \frac{1}{2}\left(\frac{3}{25}+1\right) = \frac{1}{2}\times\frac{28}{25} = \frac{14}{25}$ ✓ ← same as the answer in (a)

Check : $P[2<Y<3] = \int_2^{3} f_Y(y)\,dy = \int_2^{2.5} \frac{3}{25}\,dy + \int_{2.5}^{3} \frac{1}{5}\,dy = \frac{3}{25}\times 0.5 + \frac{1}{5}\times 0.5$

$= \frac{1}{2}\left(\frac{3}{25}+\frac{5}{25}\right) = \frac{4}{25}$ ✓ ← same as the answer in (b)

Q5: pdf, cdf, expected value, variance - current and power

Thursday, November 13, 2014   11:01 AM

$$f_X(x) = \begin{cases} 5, & 4.9 \le x \le 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) $$P[X < 5] = \int_{-\infty}^{5} f_X(x)\,dx = \cancel{\int_{-\infty}^{4.9} \underbrace{f_X(x)}_{0}\,dx}^{\,0} + \int_{4.9}^{5} \underbrace{f_X(x)}_{5}\,dx$$

$$= 5x\Big|_{4.9}^{5} = 5(5-4.9) = 5\times 0.1 = 0.5$$

(b) $$F_X(x) = P[X \le x] = \int_{-\infty}^{x} f_X(t)\,dt$$

For $x < 4.9$, $f_X(t) = 0$ for all t inside $(0, -\infty)$.

Therefore, $F_X(x) = \int_{-\infty}^{x} 0\,dt = 0$.

For $4.9 \le x \le 5.1$, $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \cancel{\int_{-\infty}^{4.9} \underbrace{f_X(t)}_{0}\,dt}^{\,0} + \int_{4.9}^{x} \underbrace{f_X(t)}_{5}\,dt$$

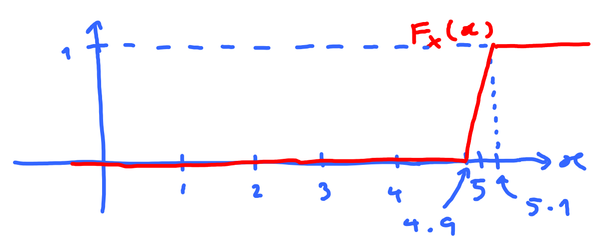
$$= 5t\Big|_{4.9}^{x} = 5\times(x-4.9) = 5x - 24.5.$$

For $x > 5.1$, $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \cancel{\int_{-\infty}^{4.9} \underbrace{f_X(t)}_{0}\,dt}^{\,0} + \int_{4.9}^{5.1} \underbrace{f_X(t)}_{5}\,dt + \cancel{\int_{5.1}^{x} \underbrace{f_X(t)}_{0}\,dt}^{\,0}$$

$$= 5t\Big|_{4.9}^{5.1} = 5\times(5.1-4.9) = 5\times 0.2 = 1.$$

Combining the three cases above, we have the complete description of the cdf:

$$F_X(x) = \begin{cases} 0, & x < 4.9, \\ 5x - 24.5, & 4.9 \le x \le 5.1, \\ 1, & x > 5.1 \end{cases}$$



Note that F_X is a continuous function. This is because it is the cdf of a continuous RV.

(c) $$\mathbb{E}X = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \cancel{\int_{0}^{4.9} x \underbrace{f_X(x)}_{0}\,dx}^{\,0} + \int_{4.9}^{5.1} x \underbrace{f_X(x)}_{5}\,dx + \cancel{\int_{5.1}^{\infty} x \underbrace{f_X(x)}_{0}\,dx}^{\,0}$$

$$= 5 \left.\frac{x^2}{2}\right|_{4.9}^{5.1} = \frac{5}{2}\left(5.1^2 - 4.9^2\right) = \frac{5}{2}(5.1+4.9)(5.1-4.9) = \frac{5}{2}(10)(0.2)$$

$$= 5 \text{ mA}$$

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\mathbb{E}X = \frac{b+a}{2} = \frac{5.1+4.9}{2} = \frac{10}{2} = 5.$

(d) $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2$. From (c), we know that $\mathbb{E}X = 5$. So, to find $\operatorname{Var} X$, we need to find $\mathbb{E}[X^2]$.

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{4.9}^{5.1} x^2 \times 5\,dx = 5 \left.\frac{x^3}{3}\right|_{4.9}^{5.1} = \frac{5}{3} \times \left(5.1^3 - 4.9^3\right)$$

$$= 25 + \frac{1}{300}.$$

Therefore, $\operatorname{Var} X = \left(25 + \frac{1}{300}\right) - 5^2 = \frac{1}{300} \approx 0.0033 \text{ (mA)}^2$

and $\sigma_X = \frac{1}{10\sqrt{3}} \text{ mA} \approx 0.0577 \text{ mA}.$

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\operatorname{Var} X = \frac{(b-a)^2}{12} = \frac{(5.1-4.9)^2}{12}$

$$= \frac{(0.2)^2}{12} = \frac{4}{100 \times 12} = \frac{1}{300}.$$

(e) Recall that $P = IV = I \times I = I^2 r$.

Here $I = X$. Therefore $P = X^2 r$ and

$$\mathbb{E}P = \mathbb{E}[X^2 r] = r\,\mathbb{E}[X^2] = 100 \times \left(25 + \frac{1}{300}\right) = 2500 + \frac{1}{3}$$

$$\approx 2.50033 \times 10^3 \left[\underbrace{(\text{mA})^2\Omega}_{\text{m}^2 \underbrace{\text{A}^2\Omega}_{\text{W}}}\right] \approx 2.5 \text{ mW}.$$

Caution: The current is in mA.

Q6: pdf and cdf - chemical reaction

Thursday, November 13, 2014 11:07 AM

$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $F_X(x)$ is a continuous function. Therefore, X is a continuous RV.

(a) $$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} -(-0.01)e^{-0.01x}, & x > 0, \\ 0, & x < 0. \end{cases} = \begin{cases} 0.01e^{-0.01x}, & x > 0, \\ 0, & x < 0. \end{cases}$$

At $x = 0$, the derivative does not exist. Because this is just a point, we may assign $f_X(0)$ to be any arbitrary value. Here, we set $f_X(0) = 0$:

$$f_X(x) = \begin{cases} 0.01e^{-0.01x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $P[X < 200] = P[X \leq 200] = F_X(200) = 1 - e^{-0.01 \times 200} = 1 - e^{-2} \approx 0.8647.$

Alternatively, $$P[X < 200] = \int_{-\infty}^{200} f_X(x)\,dx = \cancel{\int_{-\infty}^{0} f_X(x)\,dx} + \int_{0}^{200} f_X(x)\,dx$$

$$= \int_{0}^{200} 0.01e^{-0.01x}\,dx = \left.\frac{0.01}{(-0.01)}e^{-0.01x}\right|_0^{200}$$

$$= \left(-e^{-0.01 \times 200}\right) - \left(-e^{-0.01 \times 0}\right) = -e^{-2} - (-1)$$

$$= 1 - e^{-2}$$

Q7: Gaussian RV - Cholesterol

Thursday, November 13, 2014 3:49 PM

Let X be the cholesterol level of a randomly chosen adult.

It is given that $X \sim \mathcal{N}(m, \sigma^2)$ where $m = 159.2$ mg/dl.

We also know that $P[X < 200] = 0.841$.

(a) $\Phi\left(\frac{200-m}{\sigma}\right) = 0.841 \Rightarrow \frac{200-m}{\sigma} \approx 1 \Rightarrow \sigma \approx 200 - m \approx 200 - 159.2 \approx 40.8$ mg/dl

From the Φ table,

$\Phi(0.99) \approx 0.8389$

$\Phi(1) \approx 0.8413$

0.841 is closer to 0.8413 than 0.8389

(b) We find x such that $\Phi\left(\frac{x-m}{\sigma}\right) = 0.9$.

From the Φ table, $\Phi(1.28) \approx 0.8997$

$\Phi(1.29) \approx 0.9015$

0.9 is closer to 0.8997

$\Rightarrow \frac{x-m}{\sigma} \approx 1.28 \Rightarrow x \approx 1.28\underbrace{\sigma}_{\substack{40.8 \\ \text{(from (a))}}} + \underbrace{m}_{159.2} \approx 211.424$ mg/dl

(c) $P[m+\sigma < X < m+2\sigma] = F_X(m+2\sigma) - F_X(m+\sigma)$

$= \Phi\left(\frac{m+2\sigma-m}{\sigma}\right) - \Phi\left(\frac{m+\sigma-m}{\sigma}\right) = \Phi(2) - \Phi(1)$

$\approx 0.97725 - 0.8413 = 0.1359 \approx 13.59\%$

(d) $P[X > m+2\sigma] = 1 - F_X(m+2\sigma) = 1 - \Phi\left(\frac{m+2\sigma-m}{\sigma}\right) = 1 - \Phi(2)$

$\approx 1 - 0.97725 \approx 0.0228 = 2.28\%$

Q8: Exponential RV

Friday, November 21, 2014 9:02 AM

Let T be the time to failure (in hours)

We know that $T \sim \mathcal{E}(\lambda)$ where $\lambda = 3 \times 10^{-4}$.

Therefore,

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Here, we want to find $P[T > 10^4]$.

We will first provide the general formula for $P[T > t]$.

For $T \sim \mathcal{E}(\lambda)$ and $t > 0$,

$$P[T > t] = \int_t^\infty f_T(\tau)\,d\tau = \int_t^\infty \lambda e^{-\lambda \tau}\,d\tau = -e^{-\lambda \tau}\Big|_t^\infty = e^{-\lambda t}$$

Therefore, $P[T > 10^4] = e^{-3 \times 10^{-4} \times 10^4} = e^{-3} \approx 0.0498$

(b) $P[T \leq 7000] = 1 - P[T > 7000] = 1 - e^{-3 \times 10^{-4} \times 7000} = 1 - e^{-2.1} \approx 0.8775$

Remark: In class, we have already shown that for $T \sim \mathcal{E}(\lambda)$,

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$P[T > t] = \begin{cases} e^{-\lambda t}, & t > 0, \\ 1, & \text{otherwise.} \end{cases}$$

These formula can be applied here directly as well.