ECS 315: Probability and Random Processes **2014/1**

HW 11 — Due: Nov 14

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)
The extra question at the end is optional.

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Consider a random variable X whose pdf is given by

$$f_X(x) = \begin{cases} cx, & 0 < x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

(a) Find c.

(b) Find $f_X(0.2)$ and $f_X(0.8)$.

(c) Find $P[0.19 < X < 0.21]$ and $P[0.79 < X < 0.81]$.

(d) In class, we have seen that we may use $f_X(x)\Delta x$ to approximate the probability that the random variable X will be inside some small interval of length Δx near or around x.

Use this approximation to calculate $P[0.19 < X < 0.21]$ and $P[0.79 < X < 0.81]$. Compare your answers here with the exact answers in the previous part.

Problem 2. Consider a random variable X whose pdf is given by

$$f_X(x) = \begin{cases} cx^2, & 0 < x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

(a) Find c.

(b) Find $f_X(0.2)$ and $f_X(0.8)$.

(c) Find $P[0.19 < X < 0.21]$ and $P[0.79 < X < 0.81]$.

(d) In class, we have seen that we may use $f_X(x)\Delta x$ to approximate the probability that the random variable X will be inside some small interval of length Δx near or around x.

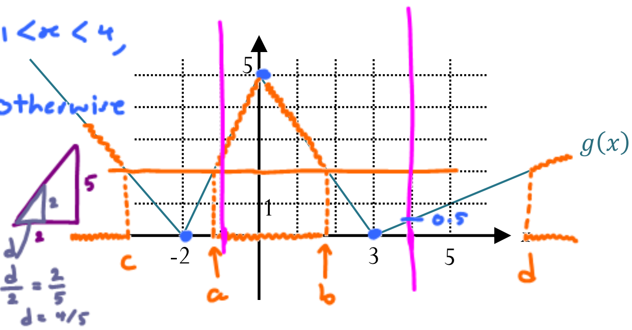
Use this approximation to calculate $P[0.19 < X < 0.21]$ and $P[0.79 < X < 0.81]$. Compare your answers here with the exact answers in the previous part.

Problem 3. Consider a random variable X whose pdf $f_X(x)$ is unknown. From an experiment, suppose we found that $P[0.99 < X < 1.03] \approx 0.1$ and $P[1.98 < X < 2.01] \approx 0.05$. Use this information to approximate the ratio $\frac{f_X(2)}{f_X(1)}$.

Problem 4. Suppose $X \sim \mathcal{U}(-1, 4)$. Define a new random variable Y by $Y = g(X)$ where the function $g(x)$ is plotted in Figure 11.1.

$$f_X(x) = \begin{cases} 1/5, & -1 < x < 4, \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1/5}{5/2} + \frac{1/5}{5/3} & 2.5 < y < 5 \\ (1/5)/(5/3) & 0.5 < y \le 2.5 \\ \frac{1/5}{5/3} + \frac{1/5}{1/2} & 0 < y < 0.5 \\ 0, & \text{otherwise} \end{cases}$$



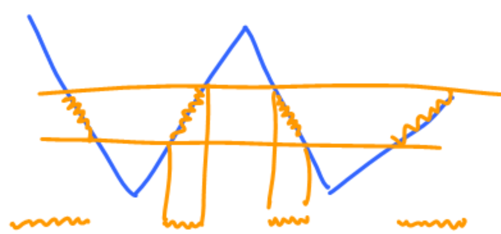


Figure 11.1: Function $g(x)$ for Problem 4.

(a) Find $P[Y > 2]$. $= P[X < c] + P[a < X < b] + P[X > d] = \cancel{\int_{-\infty}^{c} f_X(x)dx}^{\,0} + \int_{-1}^{b} \overset{1/5}{\cancel{f_X(x)}} dx + \cancel{\int_{d}^{\infty} f_X(x)dx}^{\,0}$

$= \frac{1}{5} \times (b - (-1))$

(b) Find $P[2 < Y < 3]$.

(c) Find $P[1.9 < Y < 2.1]$.

(d) Find $f_Y(y)$ at y near $y = 2$.

$$f_Y(2) = \frac{\overset{0}{\cancel{f_X(a)}}}{|5/2|} + \frac{\overset{1/5}{f_X(b)}}{|-5/3|} + \frac{\overset{0}{\cancel{f_X(c)}}}{|g'(c)|} + \frac{\overset{0}{\cancel{f_X(d)}}}{|g'(d)|}$$

(e) Find $f_Y(y)$.

$$f_Y(y) = \sum_{\substack{x \\ g(x)=y}} \frac{f_X(x)}{|g'(x)|}$$

Problem 5. Let a continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the probability density function of X is

$$f_X(x) = \begin{cases} 5, & 4.9 \le x \le 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

$P[X < 5] = \int_{-\infty}^{5} f_X(x)\,dx$

(a) Find the probability that a current measurement is less than 5 milliamperes.

(b) Find and plot the cumulative distribution function of the random variable X.

$= \int_{4.9}^{5} 5\,dx$

(c) Find the expected value of X. $\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx$

(d) Find the variance and the standard deviation of X.

(e) Find the expected value of power when the resistance is 100 ohms?

$\mathbb{E}[X^2] - (\mathbb{E}X)^2$

$P = X^2 r$

$\mathbb{E}[P] = \mathbb{E}[X^2 r] = r\,\mathbb{E}[X^2]$

Extra Question

Here are optional questions for those who want extra practice.

Problem 6. The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the probability density function of X.

(b) What proportion of reactions is complete within 200 milliseconds?

Problem 7. Cholesterol is a fatty substance that is an important part of the outer lining (membrane) of cells in the body of animals. Its normal range for an adult is 120–240 mg/dl. The Food and Nutrition Institute of the Philippines found that the total cholesterol level for Filipino adults has a mean of 159.2 mg/dl and 84.1% of adults have a cholesterol level below 200 mg/dl. Suppose that the cholesterol level in the population is normally distributed.

(a) Determine the standard deviation of this distribution.

(b) What is the value of the cholesterol level that exceeds 90% of the population?

(c) An adult is at moderate risk if cholesterol level is more than one but less than two standard deviations above the mean. What percentage of the population is at moderate risk according to this criterion?

(d) An adult is thought to be at high risk if his cholesterol level is more than two standard deviations above the mean. What percentage of the population is at high risk?

Problem 8. Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

[Montgomery and Runger, 2010, Q4-97]