ECS 315: Probability and Random Processes **2014/1**

HW Solution 10 — Due: Nov 20

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)
The extra question at the end is optional.

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Consider each random variable X defined below. Let $Y = 1 + 2X$. (i) Find and sketch the pdf of Y and (ii) Does Y belong to any of the (popular) families discussed in class? If so, state the name of the family and find the corresponding parameters.

(a) $X \sim \mathcal{U}(0, 1)$

(b) $X \sim \mathcal{E}(1)$

(c) $X \sim \mathcal{N}(0, 1)$

Solution: See handwritten solution

Problem 2. Consider each random variable X defined below. Let $Y = 1 - 2X$. (i) Find and sketch the pdf of Y and (ii) Does Y belong to any of the (popular) families discussed in class? If so, state the name of the family and find the corresponding parameters.

(a) $X \sim \mathcal{U}(0, 1)$

(b) $X \sim \mathcal{E}(1)$

(c) $X \sim \mathcal{N}(0, 1)$

Solution: See handwritten solution

Problem 3. Let $X \sim \mathcal{E}(5)$ and $Y = 2/X$. Find

(a) $F_Y(y)$.

(b) $f_Y(y)$.

(c) $\mathbb{E}Y$

Hint: Because $\frac{d}{dy}e^{-\frac{10}{y}} = \frac{10}{y^2}e^{-\frac{10}{y}} > 0$ for $y \neq 0$. We know that $e^{-\frac{10}{y}}$ is an increasing function on our range of integration. In particular, consider $y > 10/\ln(2)$. Then, $e^{-\frac{10}{y}} > \frac{1}{2}$. Hence,

$$\int_0^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}\frac{1}{2}dy = \int_{10/\ln 2}^\infty \frac{5}{y}dy.$$

Remark: To be technically correct, we should be a little more careful when writing $Y = \frac{2}{X}$ because it is undefined when $X = 0$. Of course, this happens with 0 probability; so it won't create any serious problem. However, to avoid the problem, we may define Y by

$$Y = \begin{cases} {}^2\!/_X, & X \neq 0, \\ 0, & X = 0. \end{cases}$$

Solution:

(a) We consider two cases: "$y \leq 0$" and "$y > 0$".

(i) Because $X > 0$, we know that $Y = \frac{2}{X}$ must be > 0 and hence, $F_Y(y) = 0$ for $y \leq 0$. Note that Y can not be $= 0$. We need $X = \infty$ or $-\infty$ to make $Y = 0$. However, $\pm\infty$ are not real numbers therefore they are not possible X values.

(ii) For $y > 0$,

$$F_Y(y) = P[Y \leq y] = P\left[\frac{2}{X} \leq y\right] = P\left[X \geq \frac{2}{y}\right].$$

Note that, for the last equality, we can freely move X and y without worrying about "flipping the inequality" or "division by zero" because both X and y considered here are strictly positive. Now, for $X \sim \mathcal{E}(\lambda)$ and $x > 0$, we have

$$P[X \geq x] = \int_x^\infty \lambda e^{-\lambda t}dt = -e^{-\lambda t}\Big|_x^\infty = e^{-\lambda x}$$

Therefore,

$$F_Y(y) = e^{-5\left(\frac{2}{y}\right)} = e^{\frac{-10}{y}}.$$

Combining the two cases above we have

$$F_Y(y) = \boxed{\begin{cases} e^{-\frac{10}{y}}, & y > 0 \\ 0, & y \le 0 \end{cases}}$$

(b) We show two methods that can be applied here to find $f_Y(y)$.

Method 1: Because we have already derive the cdf in the previous part, we can find the pdf via the cdf by $f_Y(y) = \frac{d}{dy}F_Y(y)$. This gives f_Y at all points except at $y = 0$ which we will set f_Y to be 0 there. Hence,

$$f_Y(y) = \boxed{\begin{cases} \frac{10}{y^2} e^{-\frac{10}{y}}, & y > 0 \\ 0, & y \le 0. \end{cases}}$$

Method 2: See handwritten solution.

(c) We can find $\mathbb{E}Y$ from $f_Y(y)$ found in the previous part or we can even use $f_X(x)$

Method 1:

$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y) = \int_0^\infty y \frac{10}{y^2} e^{-\frac{10}{y}} dy = \int_0^\infty \frac{10}{y} e^{-\frac{10}{y}} dy$$

From the hint, we have

$$\begin{aligned} \mathbb{E}Y &> \int_{10/\ln 2}^{\infty} \frac{10}{y} e^{-\frac{10}{y}} dy > \int_{10/\ln 2}^{\infty} \frac{10}{y}\frac{1}{2} dy = \int_{10/\ln 2}^{\infty} \frac{5}{y} dy \\ &= 5 \ln y \big|_{10/\ln 2}^{\infty} = \infty. \end{aligned}$$

Therefore, $\mathbb{E}Y = \boxed{\infty}$.

Method 2:

$$\begin{aligned} \mathbb{E}Y = \mathbb{E}\left[\frac{1}{X}\right] &= \int_{-\infty}^{\infty} \frac{1}{x} f_X(x)\, dx = \int_0^\infty \frac{1}{x} \lambda e^{-\lambda x} dx > \int_0^1 \frac{1}{x} \lambda e^{-\lambda x} dx \\ &> \int_0^1 \frac{1}{x} \lambda e^{-\lambda} dx = \lambda e^{-\lambda} \int_0^1 \frac{1}{x} dx = \lambda e^{-\lambda} \ln x \big|_0^1 = \infty, \end{aligned}$$

where the second inequality above comes from the fact that for $x \in (0, 1)$, $e^{-\lambda x} > e^{-\lambda}$.

Problem 4. In wireless communications systems, fading is sometimes modeled by ***lognormal*** random variables. We say that a positive random variable Y is lognormal if $\ln Y$ is a normal random variable (say, with expected value m and variance σ^2). Find the pdf of Y.

Hint: First, recall that the ln is the natural log function (log base e). Let $X = \ln Y$. Then, because Y is lognormal, we know that $X \sim \mathcal{N}(m, \sigma^2)$. Next, write Y as a function of X.

Solution:

We now show two methods that can be applied here to find $f_Y(y)$:

Method 1: Finding $f_Y(y)$ from $F_Y(y)$.

Because $X = \ln(Y)$, we have $Y = e^X$. We know that exponential function gives strictly positive number. So, Y is always strictly positive. In particular, $F_Y(y) = 0$ for $y \le 0$.

Next, for $y > 0$, by definition, $F_Y(y) = P[Y \le y]$. Plugging in $Y = e^X$, we have

$$F_Y(y) = P\left[e^X \le y\right].$$

Because the exponential function is strictly increasing, the event $[e^X \le y]$ is the same as the event $[X \le \ln y]$. Therefore,

$$F_Y(y) = P[X \le \ln y] = F_X(\ln y).$$

Combining the two cases above, we have

$$F_Y(y) = \begin{cases} F_X(\ln y), & y > 0, \\ 0, & y \le 0. \end{cases}$$

Finally, we apply

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

For $y < 0$, we have $f_Y(y) = \frac{d}{dy} 0 = 0$. For $y > 0$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\ln y) = f_X(\ln y) \times \frac{d}{dy} \ln y = \frac{1}{y} f_X(\ln y). \tag{10.1}$$

Therefore,

$$f_Y(y) = \begin{cases} \frac{1}{y} f_X(\ln y), & y > 0, \\ 0, & y < 0. \end{cases}$$

At $y = 0$, if we know that Y is a continuous random variable, we can assign any value, e.g. 0, to $f_Y(0)$. Suppose this is the case. Then

$$f_Y(y) = \boxed{\begin{cases} \frac{1}{y} f_X(\ln y), & y > 0, \\ 0, & \text{otherwise}. \end{cases}}$$

Method 2: Finding $f_Y(y)$ directly from $f_X(x)$.

Here we have $Y = g(X)$ where $g(x) = e^x$.

Note that exponential function always gives positive number. So, for $y \le 0$, there is no (real-valued) x that satisfies $g(x) = y$. Therefore, $f_Y(y) = 0$ for $y \le 0$.

For $y > 0$, there is exactly one x that satisfies $g(x) = y$, namely $x = \ln y$. At $x = \ln y$, the slope $g'(x)$ is $e^x|_{x=\ln(y)} = e^{\ln y} = y$. So,

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|} = \frac{f_X(\ln(y))}{y} = \frac{1}{y} f_X(\ln(y)).$$

Combining the two cases above, we have

$$f_Y(y) = \boxed{\begin{cases} \frac{1}{y} f_X(\ln y), & y > 0, \\ 0, & \text{otherwise.} \end{cases}}$$

Here, $X \sim \mathcal{N}(m, \sigma^2)$. Therefore,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

and

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{1}{2}\left(\frac{\ln(y)-m}{\sigma}\right)^2}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}}$$

Extra Question

Here is an optional question for those who want extra practice.

Problem 5. Consider a random variable X whose pdf is given by

$$f_X(x) = \begin{cases} cx^2, & x \in (1, 2), \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = 4|X - 1.5|$.

(a) Find $\mathbb{E}Y$.

(b) Find $f_Y(y)$.

Solution: See handwritten solution

Q1

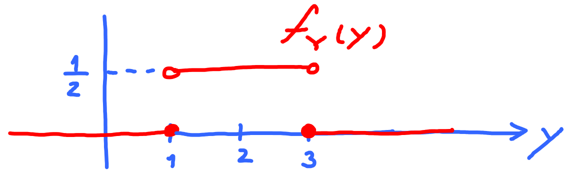
Tuesday, November 11, 2014 4:15 PM

We know that when $Y = aX+b$, we have $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$.

Here, $Y = \underset{a}{2}X + \underset{b}{1}$. Therefore, $f_Y(y) = \frac{1}{2} f_X\left(\frac{y-1}{2}\right)$.

(a) $X \sim \mathcal{U}(0,1) \Rightarrow f_X(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$

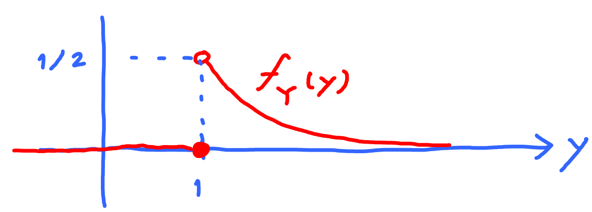
(i)
Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} 1, & 0 < \frac{y-1}{2} < 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1/2, & 1 < y < 3, \\ 0, & \text{otherwise.} \end{cases}$



(ii) Yes. $Y \sim \mathcal{U}(1,3)$

(b) $X \sim \mathcal{E}(1) \Rightarrow f_X(x) = \begin{cases} 1 \times e^{-1 \times x}, & x > 0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$

(i)
Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} e^{-\left(\frac{y-1}{2}\right)}, & \frac{y-1}{2} > 0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2}\sqrt{e}\, e^{-y/2}, & y > 1, \\ 0, & \text{otherwise.} \end{cases}$



(ii) No. Although f_Y decays exponentially, the "exponential part" starts @ $y=1$ (not @ $y=0$). We may call this distribution a shifted exponential distribution. This distribution is quite useful for modeling output of a biological neuron with refractory period.

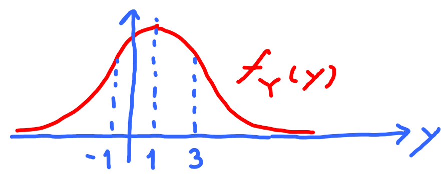
(c) We know that $X \sim \mathcal{N}(m, \sigma^2) \Rightarrow Y = \overset{2}{a}X + \overset{1}{b} \sim \mathcal{N}(am+b, a^2\sigma^2)$.

Plugging in $a=2$ and $b=1$, we have $Y \sim \mathcal{N}(2m+1, 4\sigma^2)$.

Plugging in $a=2$ and $b=1$, we have $Y \sim \mathcal{N}(2m+1, 4\sigma^2)$.

Here, $X \sim \mathcal{N}(\underbrace{0}_{m}, \underbrace{1}_{\sigma^2})$. So, $Y \sim \mathcal{N}(2\times 0+1, 4\times 1) = \mathcal{N}(1, \underbrace{4}_{\sigma = 2})$

(i) $f_Y(y) = \dfrac{1}{\sqrt{2\pi}\,2} e^{-\frac{1}{2}\left(\frac{y-1}{2}\right)^2}$



(ii) Yes. $Y \sim \mathcal{N}(1, 4)$

Q2

Tuesday, November 11, 2014 4:15 PM

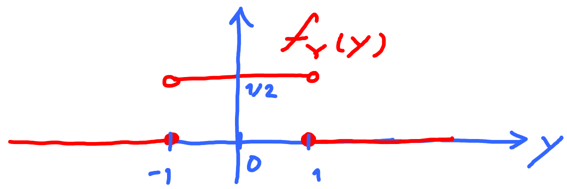
We know that when $Y = aX + b$, we have $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$.

Here, $Y = \underbrace{-2}_{a}X + \underbrace{1}_{b}$. Therefore, $f_Y(y) = \frac{1}{2} f_X\left(\frac{1-y}{2}\right)$.

(a) $X \sim \mathcal{U}(0,1) \Rightarrow f_X(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$

(i)

Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} 1, & 0 < \frac{1-y}{2} < 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1/2, & -1 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$

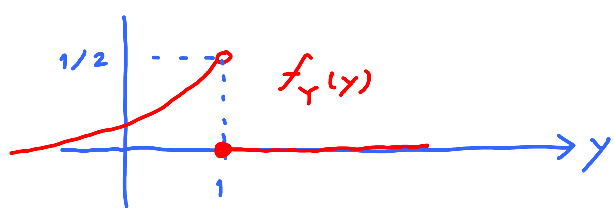


(ii) Yes. $Y \sim \mathcal{U}(-1, 1)$

(b) $X \sim \mathcal{E}(1) \Rightarrow f_X(x) = \begin{cases} 1 \times e^{-1 \times x}, & x > 0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$

(i)

Therefore, $f_Y(y) = \frac{1}{2} \times \begin{cases} e^{-\left(\frac{1-y}{2}\right)}, & \frac{1-y}{2} > 0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2\sqrt{e}} e^{y/2}, & y < 1, \\ 0, & \text{otherwise.} \end{cases}$



(ii) No. Although f_Y has exponential decay, its expression can not be rewritten in the form $f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$

(c) We know that $X \sim \mathcal{N}(m, \sigma^2) \Rightarrow Y = aX + b \sim \mathcal{N}(am + b, a^2\sigma^2)$.

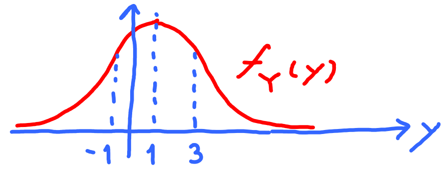
Plugging in $a = -2$ and $b = 1$, we have $Y \sim \mathcal{N}(1 - 2m, 4\sigma^2)$

Here, $X \sim \mathcal{N}(0,1)$, so $Y \sim \mathcal{N}(1 - 2 \times 0, 4 \times 1) = \mathcal{N}(1, 4)$

Plugging in $a = -2$ and $b = 1$, we have $Y \sim \mathcal{N}(1-2m, 4\sigma^2)$

Here, $X \sim \mathcal{N}(\underset{m}{0}, \underset{\sigma^2}{1})$. So, $Y \sim \mathcal{N}(-2\times 0+1, 4\times 1) = \mathcal{N}(1, 4)$ $\to \sigma = 2$

(i) $$f_Y(y) = \frac{1}{\sqrt{2\pi}\,2} e^{-\frac{1}{2}\left(\frac{y-1}{2}\right)^2}$$



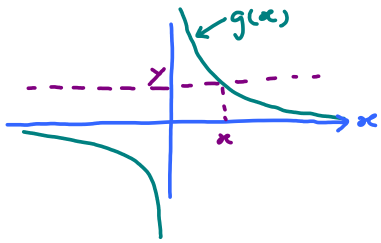
(ii) Yes. $Y \sim \mathcal{N}(1, 4)$

Q3b SISO (Method 2)

Friday, November 21, 2014   9:52 AM

(b) Let's derive a general formula for finding $f_Y(y)$ when $Y = \frac{a}{X}$ for continuous RV X and positive constant a.

Step ① Find all x that satisfy $y = g(x)$. Here, $g(x) = \frac{a}{x}$.



From the plot, we see that...

(i) For $y = 0$, there is no x that satisfies $y = g(x)$.
So, we have $f_Y(y) = 0$.

(ii) For $y \neq 0$, there is only one x that satisfies $y = g(x)$:
$$x = \frac{a}{y}.$$

Step ② The slope at this x is
$$g'\left(\frac{a}{y}\right) = -\frac{a}{x^2}\bigg|_{x=\frac{a}{y}} = -\frac{a}{\frac{a^2}{y^2}} = -\frac{y^2}{a}$$

Therefore,

Step ③
$$f_Y(y) = \frac{f_X\left(\frac{a}{y}\right)}{\left|g'\left(\frac{a}{y}\right)\right|} = \frac{a}{y^2} f_X\left(\frac{a}{y}\right).$$

Combining the cases "$y = 0$" and "$y \neq 0$", we have
$$f_Y(y) = \begin{cases} \frac{a}{y^2} f_X\left(\frac{a}{y}\right), & y \neq 0, \\ 0, & y = 0. \end{cases}$$

Now, for the problem under consideration, $X \sim \mathcal{E}(\lambda)$.

So, $$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Substituting this into $f_Y(y)$ for the case when $y \neq 0$, we have
$$\begin{cases} \lambda e^{-\lambda \times a/y}, & \frac{a}{y} \neq 0, \end{cases}$$

Substituting this into $f_Y(y)$ for the case when $y>0$, we have

$$f_Y(y) = \frac{a}{y^2} \times \begin{cases} \lambda e^{-\lambda \times a/y}, & \frac{a}{y} \neq 0, \\ 0, & \frac{a}{y} = 0. \end{cases}$$

← This can't happen when $y \neq 0$.

$$= \frac{a\lambda}{y^2} e^{-\lambda a/y} = \frac{10}{y^2} e^{-10/y}$$

$a = 2$
$\lambda = 5$

Conclusion:

$$f_Y(y) = \begin{cases} \frac{10}{y^2} e^{-10/y}, & y > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Q5

Saturday, November 22, 2014 11:54 PM

First, we need to find the constant c.

For any pdf, we know that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$

Therefore, $\int_1^2 c x^2\,dx = c\int_1^2 x^2\,dx = c\left.\frac{x^3}{3}\right|_1^2 = c\left(\frac{8-1}{3}\right) = c\times\frac{7}{3}$ must $=1$.

Hence, $c = 3/7$.

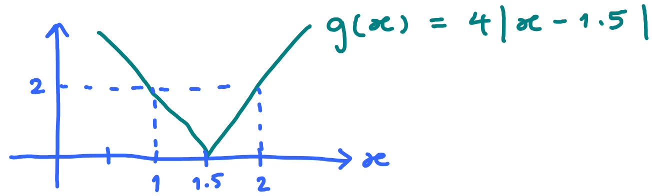
(a) $\mathbb{E}Y = \mathbb{E}\left[4|X-1.5|\right] = 4\int_1^2 |x-1.5|\,\frac{3}{7}x^2\,dx = \frac{12}{7}\int_1^2 |x-1.5|\,x^2\,dx$

$|x-1.5| = \begin{cases} x-1.5, & x \geq 1.5 \\ 1.5-x, & x < 1.5 \end{cases}$

$= \frac{12}{7}\left(\int_1^{1.5}(1.5-x)\,x^2\,dx + \int_{1.5}^{2}(x-1.5)\,x^2\,dx\right) = \frac{57}{56}$

(b) $Y = 4|X-1.5| = \left.\begin{cases} 4X-6, & X \geq 1.5, \\ 6-4X, & X < 1.5 \end{cases}\right\} \equiv g(X)$

Let's plot the function $g(x)$:



Method 1: Derive the pdf by $\frac{d}{dy}F_Y(y)$

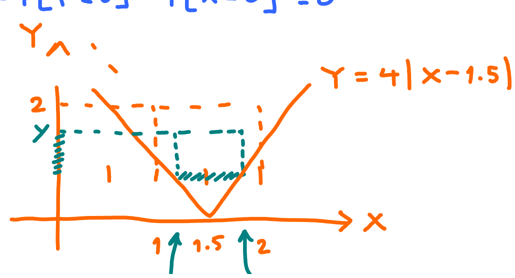
① By construction (from $|\cdot|$), we know that $Y \geq 0$. Therefore,

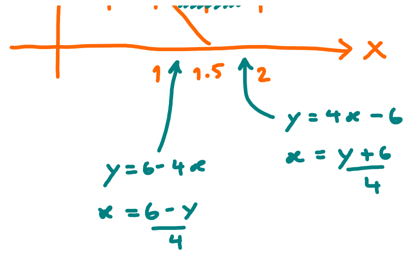
$F_Y(y) = 0$ for $y < 0$.

This means $f_Y(y) = 0$ for $y < 0$. (*)

② For $y = 0$, $F_Y(0) = P[Y \leq 0] = P[X = 0] \overset{\text{for cont. } X}{=} 0$ (**)

③ For $y > 0$,





the event $[Y \le y]$ is the same as the event $\left[\frac{6-y}{4} \le X \le \frac{6+y}{4}\right]$.

Therefore,

for cont. X

$$F_Y(y) = P\left[\frac{6-y}{4} \le X \le \frac{6+y}{4}\right] = F_X\left(\frac{6+y}{4}\right) - F_X\left(\frac{6-y}{4}\right) \quad \text{when } y > 0.$$

This implies

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{4} f_X\left(\frac{6+y}{4}\right) + \frac{1}{4} f_X\left(\frac{6-y}{4}\right) \quad \text{when } y > 0. \quad (***)$$

Plug-in $f_X(\cdot) = \frac{3}{7}(\cdot)^2$ when $1 < (\cdot) < 2$.

$$1 < \frac{6+y}{4} < 2 \qquad 1 < \frac{6-y}{4} < 2$$

$$4 < 6+y < 8 \qquad 4 < 6-y < 8$$

$$-2 < y < 2 \qquad -2 < -y < 2$$

$$-2 < y < 2$$

Note again that this analysis is valid only for $y > 0$.

Therefore,

$$f_Y(y) = \begin{cases} \frac{1}{4} \times \frac{3}{7}\left(\left(\frac{6+y}{4}\right)^2 + \left(\frac{6-y}{4}\right)^2\right), & 0 < y < 2 \\ 0, & y \ge 2 \end{cases}$$

Combining ① and ③, we have

$$f_Y(y) = \begin{cases} \frac{1}{4} \times \frac{3}{7}\left(\left(\frac{6+y}{4}\right)^2 + \left(\frac{6-y}{4}\right)^2\right), & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{3}{224}\left(y^2 + 36\right), & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

At $y = 0$, we set $f_Y(0) = 0$. This is possible because there is no jump at $y = 0$ in the cdf. ($P[Y=0] = P[X=0] = 0$.) Had there been a jump at $y = 0$, we would have had a mixed-type RV as the ans. In such case, we will deal with the jump by δ-"function."

Check $\quad \mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{0}^{2} \frac{3}{224}\left(y^3 + 36y\right) dy = \frac{57}{56}$ ← same as part (a).

Check $\mathbb{E}Y = \int_{-\infty} y f_Y(y)\,dy = \int_0 224(\ldots)\,dy = 56$ = same as part ...

Method 2. Derive the pdf of Y directly from pdf of X.

① Find x values which satisfy $y = g(x)$. There are three cases.

(i) When $y < 0$, there is no x which satisfies $y = g(x)$.

Hence, $f_Y(y) = 0$. ← same as (*) above.

(ii) When $y = 0$, there is one x value which satisfies $y = g(x)$, namely, $x = 0$.

(iii) When $y > 0$, there is two x values which satisfy $y = g(x)$, namely, $x = \frac{6 \pm y}{4}$

② Find slope at each solution of $y = g(x)$.

(ii) At $x = 0$, the slope does not exist.
However, note that $P[Y=0] = P[X=0] = 0$. ← X is a cont. RV.
Therefore, we may define $f_Y(y)$ @ $y=0$ to be any arbitrary value.
Here, we set $f_Y(0) = 0$. ← same as (**) above.

(iii) At $x = \frac{6 \pm y}{4}$, the slopes are ∓ 4, respectively.

③ $f_Y(y) = \sum_x \frac{f_X(x)}{|g'(x)|}$ where the x's are the solutions of $y = g(x)$.

(iii) When $y > 0$, $f_Y(y) = \dfrac{f_X\left(\frac{6+y}{4}\right)}{|-4|} + \dfrac{f_X\left(\frac{6-y}{4}\right)}{4}$

$$= \frac{1}{4} f_X\left(\frac{6+y}{4}\right) + \frac{1}{4}\left(\frac{6-y}{4}\right)$$ ← same as (***) above.