ECS 315: Probability and Random Processes **2014/1**

HW 10 — Due: Nov 20

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)
The extra question at the end is optional.

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

$aX+b \rightarrow a=2,\ b=1$

Problem 1. Consider each random variable X defined below. Let $Y = 1 + 2X$. (i) Find and sketch the pdf of Y and (ii) Does Y belong to any of the (popular) families discussed in class? If so, state the name of the family and find the corresponding parameters.

(a) $X \sim \mathcal{U}(0, 1)$ $\quad f_X(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise}. \end{cases}$ $\quad f_Y(y) = \frac{1}{2} f_X\left(\frac{y-1}{2}\right) = \frac{1}{2}\begin{cases} 1, & 0 < \frac{y-1}{2} < 1, \\ 0, & \text{otherwise}. \end{cases} = \begin{cases} 1/2, & 1 < y < 3 \\ 0, & \text{otherwise} \end{cases}$

(b) $X \sim \mathcal{E}(1)$ $\quad f_X(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & \text{otherwise} \end{cases}$

(c) $X \sim \mathcal{N}(0, 1)$ $\quad f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

Problem 2. Consider each random variable X defined below. Let $Y = 1 - 2X$. (i) Find and sketch the pdf of Y and (ii) Does Y belong to any of the (popular) families discussed in class? If so, state the name of the family and find the corresponding parameters.

(a) $X \sim \mathcal{U}(0, 1)$

(b) $X \sim \mathcal{E}(1)$

(c) $X \sim \mathcal{N}(0, 1)$

$$P[X > x] = 1 - F_X(x) = \begin{cases} e^{-\lambda x}, & x > 0 \\ 1, & x \le 0 \end{cases} = \begin{cases} e^{-5x}, & x > 0 \\ 1, & x \le 0 \end{cases}$$

Problem 3. Let $X \sim \mathcal{E}(5)$ and $Y = 2/X$. Find

λ

$$f_X(x) = \begin{cases} 5e^{-5x}, & x > 0, \\ 0, & \text{otherwise}. \end{cases}$$

$F_Y(y) = P[Y \le y] = 0, \quad y < 0$ don't need; x is cont.

$y > 0$

$F_Y(y) = P\left[\frac{2}{X} \le y\right] = P\left[\frac{2}{y} \le X\right]$

$= 1 - P\left[X \le 2/y\right] = 1 - F_X\left(\frac{2}{y}\right)$

$= e^{-10/y}, \quad y > 0$

(a) $F_Y(y)$.

(b) $f_Y(y)$.

$= \frac{d}{dy} F_Y(y)$

$= \begin{cases} e^{-10/y}(-10)\left(-\frac{1}{y^2}\right) = \frac{10}{y^2} e^{-10/y}, & y > 0 \end{cases}$

(c) $\mathbb{E}Y$

Hint: Because $\frac{d}{dy} e^{-\frac{10}{y}} = \frac{10}{y^2} e^{-\frac{10}{y}} > 0$ for $y \neq 0$. We know that $e^{-\frac{10}{y}}$ is an increasing function on our range of integration. In particular, consider $y > 10/\ln(2)$. Then, $e^{-\frac{10}{y}} > \frac{1}{2}$. Hence,

$$\int_0^\infty \frac{10}{y} e^{-\frac{10}{y}} dy > \int_{10/\ln 2}^\infty \frac{10}{y} e^{-\frac{10}{y}} dy > \int_{10/\ln 2}^\infty \frac{10}{y} \frac{1}{2} dy = \int_{10/\ln 2}^\infty \frac{5}{y} dy.$$

Remark: To be technically correct, we should be a little more careful when writing $Y = \frac{2}{X}$ because it is undefined when $X = 0$. Of course, this happens with 0 probability; so it won't create any serious problem. However, to avoid the problem, we may define Y by

$$Y = \begin{cases} 2/X, & X \neq 0, \\ 0, & X = 0. \end{cases}$$

Problem 4. In wireless communications systems, fading is sometimes modeled by ***lognormal*** random variables. We say that a positive random variable Y is lognormal if $\ln Y$ is a normal random variable (say, with expected value m and variance σ^2). Find the pdf of Y.

Hint: First, recall that the ln is the natural log function (log base e). Let $X = \ln Y$. Then, because Y is lognormal, we know that $X \sim \mathcal{N}(m, \sigma^2)$. Next, write Y as a function of X.

$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$

$X = \ln Y \sim \mathcal{N}(m, \sigma^2)$

$Y = e^X \rightarrow g(x) = e^x \quad g'(x) = e^x$

$$f_Y(y) = \begin{cases} \frac{f_X(\ln y)}{|y|}, & y > 0 \\ 0, & \text{otherwise}. \end{cases}$$

Extra Question

Here is an optional question for those who want extra practice.

Problem 5. Consider a random variable X whose pdf is given by

$$f_X(x) = \begin{cases} cx^2, & x \in (1, 2), \\ 0, & \text{otherwise}. \end{cases}$$

Let $Y = 4|X - 1.5|$.

(a) Find $\mathbb{E}Y$.

(b) Find $f_Y(y)$.