ECS 315: Probability and Random Processes **2014/1**

HW 1 — Due: Aug 28

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

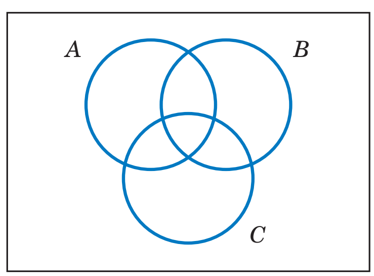
(b) It is important that you try to solve all problems. (5 pt)
The extra questions at the end are optional.

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. (Set Theory)

(a) Three events are shown on the Venn diagram in the following figure:



Reproduce the figure and shade the region that corresponds to each of the following events.

(i) A^c

(ii) $A \cap B$

(iii) $(A \cap B) \cup C$

(iv) $(B \cup C)^c$

(v) $(A \cap B)^c \cup C$

[Montgomery and Runger, 2010, Q2-19]

(b) Let $\Omega = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and put $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{5, 6\}$. Find $A \cup B$, $A \cap B$, $A \cap C$, A^c, and $B \setminus A$.

For this problem, only answers are needed; you don't have to describe your solution.

Problem 2. (**Classical Probability and Combinatorics**) A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases.

(a) How many different designs are possible? [Montgomery and Runger, 2010, Q2-51]

(b) A specific design is randomly generated by the Web server when you visit the site. If you visit the site five times, what is the probability that you will not see the same design? [Montgomery and Runger, 2010, Q2-71]

Problem 3. (**Classical Probability and Combinatorics**) A bin of 50 parts contains five that are defective. A sample of two parts is selected at random, without replacement. Determine the probability that both parts in the sample are defective. [Montgomery and Runger, 2010, Q2-49]

Problem 4. (**Classical Probability and Combinatorics**) We all know that the chance of a head (H) or tail (T) coming down after a fair coin is tossed are fifty-fifty. If a fair coin is tossed ten times, then intuition says that five heads are likely to turn up.

Calculate the probability of getting exactly five heads (and hence exactly five tails).

Problem 5. (**Classical Probability and Combinatorics**) Shuffle a deck of cards and cut it into three piles. What is the probability that (at least) a court card will turn up on top of one of the piles.

Hint: There are 12 court cards (four jacks, four queens and four kings) in the deck.

$$\frac{|A|}{|\Omega|} = \frac{|\Omega| - |A^c|}{52 \times 51 \times 50} = \frac{52 \times 51 \times 50 - 40 \times 39 \times 38}{52 \times 51 \times 50} = 1 - \frac{40}{52} \times \frac{39}{51} \times \frac{38}{50}$$

$$\approx 0.553$$

Problem 6. (**Classical Probability**) There are three buttons which are painted red on one side and white on the other. If we tosses the buttons into the air, calculate the probability that all three come up the same color.

Remarks: A *wrong* way of thinking about this problem is to say that there are four ways they can fall. All red showing, all white showing, two reds and a white or two whites and a red. Hence, it seems that out of four possibilities, there are two favorable cases and hence the probability is 1/2.

Problem 7. Each of the possible five outcomes of a random experiment is equally likely. The sample space is $\{a, b, c, d, e\}$. Let A denote the event $\{a, b\}$, and let B denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

[Montgomery and Runger, 2010, Q2-54]

Extra Questions

Here are optional questions for those who want more practice. Caution: Some questions are challenging.

Problem 8. (**Combinatorics**) Consider the design of a communication system in the United States.

(a) How many three-digit phone prefixes that are used to represent a particular geographic area (such as an area code) can be created from the digits 0 through 9?

(b) How many three-digit phone prefixes are possible in which no digit appears more than once in each prefix?

(c) As in part (a), how many three-digit phone prefixes are possible that do not start with 0 or 1, but contain 0 or 1 as the middle digit?

[Montgomery and Runger, 2010, Q2-45]

Problem 9. ***Binomial theorem***: For any positive integer n, we know that

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r}. \tag{1.1}$$

(a) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$? $\binom{25}{12} = 5,200,300$

(b) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

$0 =$ (c) Use the binomial theorem (1.1) to evaluate $\sum_{k=0}^{n}(-1)^k\binom{n}{k}$. $x=-1, y=1$

$$\binom{25}{12}(2x)^{12}(-3y)^{13}$$

$$\binom{25}{12}2^{12}(-3)^{13} = \boxed{-\binom{25}{12}2^{12}3^{13}}$$

$$= -33959763545702400$$

(d) Use the binomial theorem (1.1) to simplify the following sums

(i) $\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r}x^r(1-x)^{n-r}$

(ii) $\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r}x^r(1-x)^{n-r}$

(e) If we differentiate (1.1) with respect to x and then multiply by x, we have

$$\sum_{r=0}^{n} r\binom{n}{r}x^r y^{n-r} = nx(x+y)^{n-1}.$$

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r}$$

Use similar technique to simplify the sum $\sum_{r=0}^{n} r^2\binom{n}{r}x^r y^{n-r}$.

$$n(x+y)^{n-1} = \sum\binom{n}{r} r x^{r-1} y^{n-r}$$

$$x\frac{d}{dx}\left(nx(x+y)^{n-1}\right) \qquad nx(x+y)^{n-1} = \sum\binom{n}{r} r x^r y^{n-r}$$

Problem 10. ***An Even Split at Coin Tossing***: Let p_n be the probability of getting exactly n heads (and hence exactly n tails) when a fair coin is tossed $2n$ times.

(a) Find p_n. $\frac{\binom{10}{5}}{2^{10}}$ $\quad \frac{\binom{2n}{n}}{2^{2n}} = \frac{(2n)!}{n!n!\,2^{2n}} \approx \frac{\cancel{4^n}}{\sqrt{\pi n}\,\cancel{2^{2n}}} = \frac{1}{\sqrt{\pi n}}$

(b) Sometimes, to work theoretically with large factorials, we use Stirling's Formula:

$e^{\ln 5} = 5$

$$n! \approx \sqrt{2\pi n}\,n^n e^{-n} = \left(\sqrt{2\pi e}\right)e^{\left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)}. \tag{1.2}$$

Approximate p_n using Stirling's Formula.

(c) Find $\lim_{n\to\infty} p_n$.

Problem 11. (**Classical Probability and Combinatorics**) Suppose ~~n~~ 5 integers are chosen with replacement (that is, the same integer could be chosen repeatedly) at random from $\{1, 2, 3, \ldots, \cancel{N}\,11\}$. Calculate the probability that the chosen numbers arise according to some non-decreasing sequence.

$$\binom{n+N-1}{n} \qquad 2\ 4\ 1\ 3\ 8 \qquad \binom{15}{5} = \binom{15}{10}$$

$$2\ 4\ 5\ 6\ 6$$

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$$x_1 + x_2 + x_3 + \cdots x_{\cancel{11}\,N} = \cancel{5}\,n$$

$$|\Omega| = 11^5$$

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

$$(x+y)^5 = \binom{5}{0} x^0 y^5 + \binom{5}{1} x^1 y^4 + \binom{5}{2} x^2 y^3 + \binom{5}{3} x^3 y^2 + \binom{5}{4} x^4 y^1 + \binom{5}{5} x^5 y^0$$

$$(x-y)^5 = - \quad + \quad - \quad + \quad - \quad +$$

$$\underbrace{\cancel{(x+y)^n}}_{1} - \underbrace{(x-y)^n}_{(2x-1)^n} = 2 \sum_{\substack{r=0 \\ r\,\text{even}}}^{n} \binom{n}{r} x^r y^{n-r}$$

$x = x$

$y = 1 - x$

$x - (1-x)$

$$\underbrace{\cancel{(x+y)^n}}_{1} + \underbrace{(x-y)^n}_{(2x-1)^n} = 2 \sum_{\substack{r=0 \\ r\,\text{odd}}}^{n} \binom{n}{r} x^r y^{n-r}$$