ECS 315: Probability and Random Processes **2014/1**

HW Solution 1 — Due: Aug 28

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

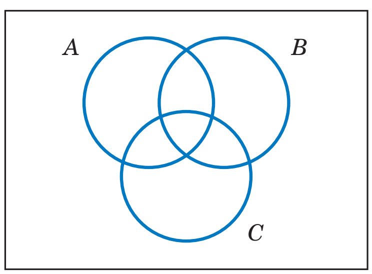
(b) It is important that you try to solve all problems. (5 pt)
The extra questions at the end are optional.

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. (**Set Theory**)

(a) Three events are shown on the Venn diagram in the following figure:



Reproduce the figure and shade the region that corresponds to each of the following events.

(i) A^c

(ii) $A \cap B$

(iii) $(A \cap B) \cup C$

(iv) $(B \cup C)^c$

(v) $(A \cap B)^c \cup C$

[Montgomery and Runger, 2010, Q2-19]

(b) Let $\Omega = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and put $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{5, 6\}$. Find $A \cup B$, $A \cap B$, $A \cap C$, A^c, and $B \setminus A$.

For this problem, only answers are needed; you don't have to describe your solution.

Solution:

(a) See Figure 1.1

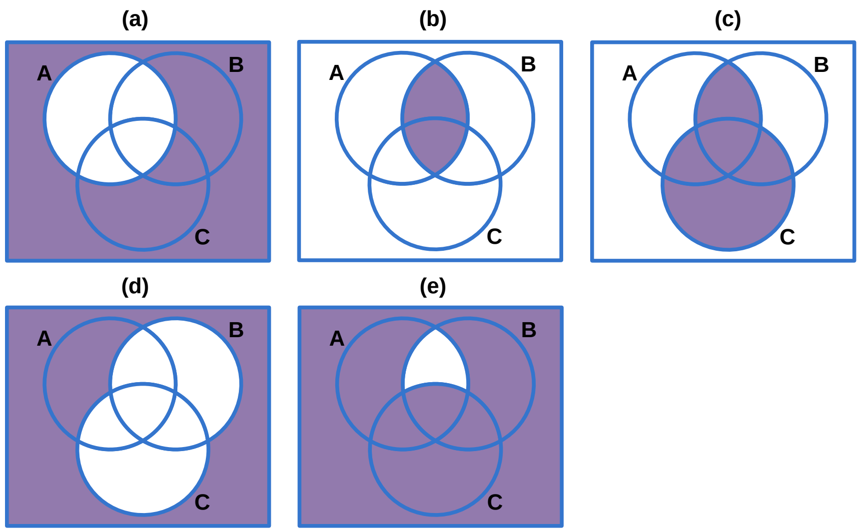


Figure 1.1: Venn diagrams for events in Problem 1

(b) $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $A \cap B = \{3, 4\}$, $A \cap C = \emptyset$, $B \setminus A = \{5, 6\} = C$.

Problem 2. (**Classical Probability and Combinatorics**) A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases.

(a) How many different designs are possible? [Montgomery and Runger, 2010, Q2-51]

(b) A specific design is randomly generated by the Web server when you visit the site. If you visit the site five times, what is the probability that you will not see the same design? [Montgomery and Runger, 2010, Q2-71]

Solution:

(a) By the multiplication rule, total number of possible designs

$$= 4 \times 3 \times 5 \times 3 \times 5 = \boxed{900}.$$

(b) From part (a), total number of possible designs is 900. The sample space is now the set of all possible designs that may be seen on five visits. It contains $(900)^5$ outcomes. (This is ordered sampling with replacement.)

The number of outcomes in which all five visits are different can be obtained by realizing that this is ordered sampling without replacement and hence there are $(900)_5$ outcomes. (Alternatively, On the first visit any one of 900 designs may be seen. On the second visit there are 899 remaining designs. On the third visit there are 898 remaining designs. On the fourth and fifth visits there are 897 and 896 remaining designs, respectively. From the multiplication rule, the number of outcomes where all designs are different is $900 \times 899 \times 898 \times 897 \times 896$.)

Therefore, the probability that a design is not seen again is

$$\frac{(900)_5}{900^5} \approx \boxed{0.9889.}$$

Problem 3. (**Classical Probability and Combinatorics**) A bin of 50 parts contains five that are defective. A sample of two parts is selected at random, without replacement. Determine the probability that both parts in the sample are defective. [Montgomery and Runger, 2010, Q2-49]

Solution: The number of ways to select two parts from 50 is $\binom{50}{2}$ and the number of ways to select two defective parts from the 5 defective ones is $\binom{5}{2}$ Therefore the probability is

$$\frac{\binom{5}{2}}{\binom{50}{2}} = \frac{2}{245} = \boxed{0.0082}.$$

Problem 4. (**Classical Probability and Combinatorics**) We all know that the chance of a head (H) or tail (T) coming down after a fair coin is tossed are fifty-fifty. If a fair coin is tossed ten times, then intuition says that five heads are likely to turn up.

Calculate the probability of getting exactly five heads (and hence exactly five tails).

Solution: There are 2^{10} possible outcomes for ten coin tosses. (For each toss, there is two possibilities, H or T). Only $\binom{10}{5}$ among these outcomes have exactly heads and five tails.

(Choose 5 positions from 10 position for H. Then, the rest of the positions are automatically T.) The probability of have exactly 5 H and 5 T is

$$\boxed{\frac{\binom{10}{5}}{2^{10}} \approx 0.246.}$$

Note that five heads and five tails will turn up more frequently than any other single combination (one head, nine tails for example) but the sum of all the other possibilities is much greater than the single 5 H, 5 T combination.

Problem 5. (**Classical Probability and Combinatorics**) Shuffle a deck of cards and cut it into three piles. What is the probability that (at least) a court card will turn up on top of one of the piles.

Hint: There are 12 court cards (four jacks, four queens and four kings) in the deck.

Solution: In [Lovell, 2006, p. 17–19], this problem is named "Three Lucky Piles". When somebody cuts three piles, they are, in effect, randomly picking three cards from the deck. There are $52 \times 51 \times 50$ possible outcomes. The number of outcomes that do not contain any court card is $40 \times 39 \times 38$. So, the probability of having at least one court card is

$$\boxed{\frac{52 \times 51 \times 50 - 40 \times 39 \times 38}{52 \times 51 \times 50} \approx 0.553.}$$

Problem 6. (**Classical Probability**) There are three buttons which are painted red on one side and white on the other. If we tosses the buttons into the air, calculate the probability that all three come up the same color.

Remarks: A *wrong* way of thinking about this problem is to say that there are four ways they can fall. All red showing, all white showing, two reds and a white or two whites and a red. Hence, it seems that out of four possibilities, there are two favorable cases and hence the probability is 1/2.

Solution: There are 8 possible outcomes. (The same number of outcomes as tossing three coins.) Among these, only two outcomes will have all three buttons come up the same color. So, the probability is $2/8 = \boxed{1/4}$.

Problem 7. Each of the possible five outcomes of a random experiment is equally likely. The sample space is $\{a, b, c, d, e\}$. Let A denote the event $\{a, b\}$, and let B denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

[Montgomery and Runger, 2010, Q2-54]

Solution: Because the outcomes are equally likely, we can simply use classical probability.

(a) $P(A) = \frac{|A|}{|\Omega|} = \boxed{\frac{2}{5}}$

(b) $P(B) = \frac{|B|}{|\Omega|} = \boxed{\frac{3}{5}}$

(c) $P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{5-2}{5} = \boxed{\frac{3}{5}}$

(d) $P(A \cup B) = \frac{|\{a,b,c,d,e\}|}{|\Omega|} = \frac{5}{5} = \boxed{1}$

(e) $P(A \cap B) = \frac{|\emptyset|}{|\Omega|} = \boxed{0}$

Extra Questions

Here are optional questions for those who want more practice. Caution: Some questions are challenging.

Problem 8. (**Combinatorics**) Consider the design of a communication system in the United States.

(a) How many three-digit phone prefixes that are used to represent a particular geographic area (such as an area code) can be created from the digits 0 through 9?

(b) How many three-digit phone prefixes are possible in which no digit appears more than once in each prefix?

(c) As in part (a), how many three-digit phone prefixes are possible that do not start with 0 or 1, but contain 0 or 1 as the middle digit?

[Montgomery and Runger, 2010, Q2-45]

Solution:

(a) From the multiplication rule (or by realizing that this is ordered sampling with replacement), $10^3 = \boxed{1,000}$ prefixes are possible

(b) This is ordered sampling without replacement. Therefore $(10)_3 = 10 \times 9 \times 8 = \boxed{720}$ prefixes are possible

(c) From the multiplication rule, $8 \times 2 \times 10 = \boxed{160}$ prefixes are possible.

Problem 9. ***Binomial theorem***: For any positive integer n, we know that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{1.1}$$

(a) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

(b) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

(c) Use the binomial theorem (1.1) to evaluate $\sum\limits_{k=0}^{n}(-1)^k\binom{n}{k}$.

(d) Use the binomial theorem (1.1) to simplify the following sums

(i) $\sum\limits_{\substack{r=0\\ r \text{ even}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

(ii) $\sum\limits_{\substack{r=0\\ r \text{ odd}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

(e) If we differentiate (1.1) with respect to x and then multiply by x, we have

$$\sum_{r=0}^{n} r\binom{n}{r} x^r y^{n-r} = nx(x+y)^{n-1}.$$

Use similar technique to simplify the sum $\sum_{r=0}^{n} r^2\binom{n}{r}x^r y^{n-r}$.

Solution:

(a) $\binom{25}{12} = \boxed{5,200,300}$.

(b) $\binom{25}{12}2^{12}(-3)^{13} = -\frac{25!}{12!13!}2^{12}3^{13} = \boxed{-33959763545702400}$.

(c) From (1.1), set $x = -1$ and $y = 1$, then we have $\sum\limits_{k=0}^{n}(-1)^k\binom{n}{k} = (-1+1)^n = \boxed{0}$.

(d) To deal with the sum involving only the even terms (or only the odd terms), we first use (1.1) to expand $(x+y)^n$ and $(x+(-y))^n$. When we add the expanded results, only the even terms in the sum are left. Similarly, when we find the difference between the two expanded results, only the the odd terms are left. More specifically,

$$\sum_{\substack{r=0\\ r \text{ even}}}^{n} \binom{n}{r} x^r y^{n-r} = \frac{1}{2}\left((x+y)^n + (y-x)^n\right), \text{ and}$$
$$\sum_{\substack{r=0\\ r \text{ odd}}}^{n} \binom{n}{r} x^r y^{n-r} = \frac{1}{2}\left((x+y)^n - (y-x)^n\right).$$

If $x + y = 1$, then

$$\sum_{\substack{r=0\\ r \text{ even}}}^{n} \binom{n}{r} x^r y^{n-r} = \boxed{\frac{1}{2}\left(1 + (1-2x)^n\right)}, \text{ and} \tag{1.2a}$$
$$\sum_{\substack{r=0\\ r \text{ odd}}}^{n} \binom{n}{r} x^r y^{n-r} = \boxed{\frac{1}{2}\left(1 - (1-2x)^n\right)}. \tag{1.2b}$$

(e) $\sum_{r=0}^{n} r^2\binom{n}{r}x^r y^{n-r} = \boxed{nx\left(x(n-1)(x+y)^{n-2} + (x+y)^{n-1}\right)}$.

Problem 10. ***An Even Split at Coin Tossing***: Let p_n be the probability of getting exactly n heads (and hence exactly n tails) when a fair coin is tossed $2n$ times.

(a) Find p_n.

(b) Sometimes, to work theoretically with large factorials, we use Stirling's Formula:

$$n! \approx \sqrt{2\pi n}\, n^n e^{-n} = \left(\sqrt{2\pi e}\right) e^{\left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)}. \tag{1.3}$$

Approximate p_n using Stirling's Formula.

(c) Find $\lim_{n\to\infty} p_n$.

Solution: Note that we have worked on a particular case ($n = 5$) of this problem earlier.

(a) Use the same solution as Problem 4; change 5 to n and 10 to $2n$, we have

$$p_n = \boxed{\frac{\binom{2n}{n}}{2^{2n}}}.$$

(b) By Stirling's Formula, we have

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} \approx \frac{\sqrt{2\pi 2n}(2n)^{2n}e^{-2n}}{\left(\sqrt{2\pi n}n^n e^{-n}\right)^2} = \frac{4^n}{\sqrt{\pi n}}.$$

Hence,

$$p_n \approx \boxed{\frac{1}{\sqrt{\pi n}}}. \tag{1.4}$$

[Mosteller, *Fifty Challenging Problems in Probability with Solutions*, 1987, Problem 18]

See Figure 1.2 for comparison of p_n and its approximation via Stirling's formula.

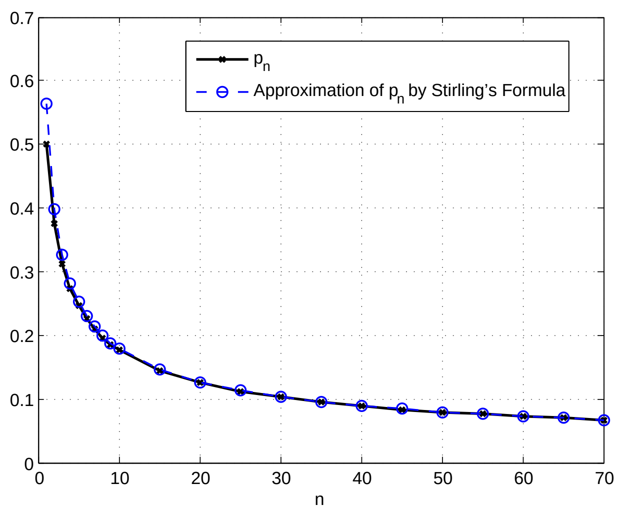


Figure 1.2: Comparison of p_n and its approximation via Stirling's formula

(c) From (1.4), $\lim_{n\to\infty} p_n = \boxed{0}$. A more rigorous proof of this limit would use the bounds

$$\frac{4^n}{\sqrt{4n}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{3n+1}}.$$

Problem 11. (**Classical Probability and Combinatorics**) Suppose n integers are chosen with replacement (that is, the same integer could be chosen repeatedly) at random from $\{1, 2, 3, \dots, N\}$. Calculate the probability that the chosen numbers arise according to some non-decreasing sequence.

Solution: There are N^n possible sequences. (This is ordered sampling with replacement.) To find the probability, we need to count the number of non-decreasing sequences among these N^n possible sequences. It takes some thought to realize that this is exactly the counting problem that we called "unordered sampling with replacement". In which case, we can conclude that the probability is $\boxed{\frac{\binom{n+N-1}{n}}{N^n}}$. The "with replacement" part should be clear from the question statement. The "unordered" part needs some more thought.

To see this, let's look back at how we turn the "ordered sampling *without replacement*" into "unordered sampling *without replacement*". Recall that there are $(N)_n$ distinct samples for "ordered sampling without replacement". When we switch to the "unordered" case, we see that many of the original samples from the "ordered sampling without replacement" are regarded as the same in the "unordered" case. In fact, we can form "groups" of samples whose members are regarded as the same in the "unordered" case. We can then count the number of groups. In class, we found that the size of any individual group can be calculated easily from permuting the elements in a sample and hence there are $n!$ members in each group. This leads us to conclude that there are $(N)_n/n! = \binom{N}{n}$ groups.

We are in a similar situation when we want to turn the "ordered sampling *with replacement*" into "unordered sampling *with replacement*". We first start with N^n distinct samples from "ordered sampling with replacement". Now, we again separate these samples into groups. Let's consider an example where $n = 3$. Then sequences "1 1 2", "1 2 1", and "2 1 1" are put together in the same group in the "unordered" case. Note that the size of this group is 3. The sequences "1 2 3", "1 3 2", "2 1 3", "2 3 1", "3 1 2", and "3 2 1" are in another group. Note that the size of this group is 6. Therefore, the group sizes are not the same and hence we can not find the number of groups by $N^n/(\text{group size})$ as in the sampling *without replacement* discussed in the previous paragraph. To count the number of groups, we look at the sequences from another perspective. We see that the "unordered" case, the only information the characterizes each group is "how many of each number there are". This is why we can match the number of groups to the number of nonnegative-integer solutions to the equation $x_1 + x_2 + \cdots + x_N = n$ as discussed in class. Finally, note that for each group, we have only one possible nondecreasing sequence. So, the number of possible nondecreasing sequence is the same as the number of groups.

If you think about the explanation above, you may realize that, by requiring the "order" on the sequence, the counting problem become "unordered sampling".

Here, we present two direct methods that leads to the same answer.

Method 1: Because the sequence is non-decreasing, the number of times that each of the integers $\{1, 2, \dots, N\}$ shows up in the sequence is the only information that characterizes each sequence. Let x_i be the number of times that number i shows up in the sequence. The number of sequences is then the same as the number of solution to the equation $x_1 + x_2 + \cdots + x_N = n$ where the x_i are all non-negative integers. We have seen in class that the number of solutions is $\binom{n+N-1}{n}$.

Method 2: [DasGupta, 2010, Example 1.14, p. 12] Consider the following construction of such non-decreasing sequence. Start with n stars and $N-1$ bars. There are $\binom{n+N-1}{n}$ arrangements of these. For example, when $N = 5$ and $n = 2$, one arrangement is $|*||*|$. Now, add spaces between these bars and stars including before the first one and after the last one. For our earlier example, we have $\_|\_*\_|\_|\_*\_|\_$. Now, put number 1 in the leftmost space. After this position, the next space holds the same value as the previous on if you pass a $*$. On the other hand, if you pass a $|$ then the value increases by 1. Note that because there are $N-1$ bars, the last space always gets the value N. What you now have is a sequence of $n+N$ numbers with bars between consecutive distinct numbers and stars between consecutive equal numbers. For example, our example would gives $1|2*2|3|4*4|5$. Note that this gives a non-decreasing sequence of $n+N$ numbers. The corresponding non-decreasing sequence of n numbers for this arrangement of stars and bars is (2,4); that is we only take the numbers to the right of the stars. Because there are n stars, our sequence will have n numbers. It will be non-decreasing because it is a sub-sequence of the non-decreasing $n+N$ sequence. This shows that any arrangement of n stars and $N-1$ bars gives one nondecreasing sequence of n numbers.

Conversely, we can take any nondecreasing sequence of n numbers and combine it with the full set of numbers $\{1, 2, 3, \dots, N\}$ to form a set of $n+N$ numbers. Now rearrange these numbers in a nondecreasing order. Put a bar between consecutive distinct numbers in this set and a star between consecutive equal numbers in this set. Note that the number to the right of each star is an element of the original n-number sequence. This shows that any nondecreasing sequence of n numbers corresponds to an arrangement of n stars and $N-1$ bars.

Combining the two paragraphs above, we now know that the number of non-decreasing sequences is the same as the number of arrangement of n stars and $N-1$ bars, which is $\binom{n+N-1}{n}$.

Remark: There is also a method— which will not be discussed here, but can be inferred by finding the pattern of the sums that lead to the number of non-decreasing sequences as we increase the value of n— that would interestingly give the number of non-decreasing sequences as

$$\sum_{k_{n-1}=1}^{N} \cdots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} k_1.$$

This sum can be simplified into $\binom{n+N-1}{n}$ by the "parallel summation formula" which is well-known but we didn't discuss in class because this is not a class on combinatorics.

ECS 315: Probability and Random Processes	2014/1
HW Solution 2 — Due: Sep 4	
Lecturer: Prapun Suksompong, Ph.D.	

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. If A, B, and C are disjoint events with $P(A) = 0.2$, $P(B) = 0.3$ and $P(C) = 0.4$, determine the following probabilities:

(a) $P(A \cup B \cup C)$

(b) $P(A \cap B \cap C)$

(c) $P(A \cap B)$

(d) $P((A \cup B) \cap C)$

(e) $P(A^c \cap B^c \cap C^c)$

[Montgomery and Runger, 2010, Q2-75]

Solution:

(a) Because A, B, and C are disjoint, $P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.3 + 0.2 + 0.4 = \boxed{0.9.}$

(b) Because A, B, and C are disjoint, $A \cap B \cap C = \emptyset$ and hence $P(A \cap B \cap C) = P(\emptyset) = \boxed{0}$.

(c) Because A and B are disjoint, $A \cap B = \emptyset$ and hence $P(A \cap B) = P(\emptyset) = \boxed{0}$.

(d) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. By the disjointness among A, B, and C, we have $(A \cap C) \cup (B \cap C) = \emptyset \cup \emptyset = \emptyset$. Therefore, $P((A \cup B) \cap C) = P(\emptyset) = \boxed{0}$.

(e) From $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$, we have $P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C) = 1 - 0.9 = \boxed{0.1.}$

Problem 2. The sample space of a random experiment is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

[Montgomery and Runger, 2010, Q2-55]

Solution:

(a) Recall that the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Therefore,

$$\begin{aligned} P(A) &= P(\{a, b, c\}) = P(\{a\}) + P(\{b\}) + P(\{c\}) \\ &= 0.1 + 0.1 + 0.2 = \boxed{0.4.} \end{aligned}$$

(b) Again, the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Thus,

$$\begin{aligned} P(B) &= P(\{c, d, e\}) = P(\{c\}) + P(\{d\}) + P(\{e\}) \\ &= 0.2 + 0.4 + 0.2 = \boxed{0.8.} \end{aligned}$$

(c) Applying the complement rule, we have $P(A^c) = 1 - P(A) = 1 - 0.4 = \boxed{0.6.}$

(d) Note that $A \cup B = \Omega$. Hence, $P(A \cup B) = P(\Omega) = \boxed{1.}$

(e) $P(A \cap B) = P(\{c\}) = \boxed{0.2.}$

Problem 3.

(a) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Find the range of possible values for $P(A \cap B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.21]

(b) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Find the range of possible values for $P(A \cup B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.22]

Solution:

(a) We will try to derive general bounds for $P(A \cap B)$.

First, recall[1], from the lecture notes, that "$P(A \cap B)$ can not exceed $P(A)$ and $P(B)$":

$$P(A \cap B) \le \min\{P(A), P(B)\}. \tag{2.1}$$

On the other hand, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{2.2}$$

Now, $P(A \cup B)$ is a probability and hence its value must be between 0 and 1:

$$0 \le P(A \cup B) \le 1 \tag{2.3}$$

Combining (2.3) with (2.2) gives

$$P(A) + P(B) - 1 \le P(A \cap B) \le P(A) + P(B). \tag{2.4}$$

The second inequality in (2.4) is not useful because (2.1) gives a better[2] bound. So, we will replace the second inequality with (2.1):

$$P(A) + P(B) - 1 \le P(A \cap B) \le \min\{P(A), P(B)\}. \tag{2.5}$$

Finally, $P(A \cap B)$ is also a probability and hence it must be between 0 and 1:

$$0 \le P(A \cap B) \le 1 \tag{2.6}$$

Combining (2.6) and (2.5), we have

$$\max\{(P(A) + P(B) - 1), 0\} \le P(A \cap B) \le \min\{P(A), P(B), 1\}.$$

[1] Again, to see this, note that $A \cap B \subset A$ and $A \cap B \subset B$. Hence, we know that $P(A \cap B) \le P(A)$ and $P(A \cap B) \le P(B)$.

[2] When we already know that a number is less than 3, learning that it is less than 5 does not give us any new information.

Note that number one at the end of the expression above is not necessary because the two probabilities under minimization can not exceed 1 themselves. In conclusion,

$$\max\{(P(A)+P(B)-1),0\} \le P(A\cap B) \le \min\{P(A),P(B)\}.$$

Plugging in the value $P(A)=\frac{1}{2}$ and $P(B)=\frac{2}{3}$ gives the range $\boxed{\left[\frac{1}{6},\frac{1}{2}\right]}$.

Note that the upper-bound can be obtained by constructing an example which has $A \subset B$. The lower-bound can be obtained by considering an example where $A \cup B = \Omega$.

(b) We will try to derive general bounds for $P(A\cup B)$.

By monotonicity, because both A and B are subset of $A\cup B$, we must have

$$P(A\cup B) \ge \max\{P(A),P(B)\}.$$

On the other hand, we know, from the finite sub-additivity property, that

$$P(A\cup B) \le P(A)+P(B).$$

Therefore,

$$\max\{P(A),P(B)\} \le P(A\cup B) \le P(A)+P(B).$$

Being a probability, $P(A\cup B)$ must be between 0 and 1. Hence,

$$\max\{P(A),P(B),0\} \le P(A\cup B) \le \min\left\{\left(P(A)+P(B)\right),1\right\}.$$

Note that number 0 is not needed in the aximization because the two probabilities involved are automatically ≥ 0 themselves.

In conclusion,

$$\max\{P(A),P(B)\} \le P(A\cup B) \le \min\{(P(A)+P(B)),1\}.$$

Plugging in the value $P(A)=\frac{1}{2}$ and $P(B)=\frac{1}{3}$, we have

$$P(A\cup B) \in \boxed{\left[\frac{1}{2},\frac{5}{6}\right]}.$$

The upper-bound can be obtained by making $A \perp B$. The lower-bound is achieved when $B \subset A$.

Problem 4. Let A and B be events for which $P(A)$, $P(B)$, and $P(A \cup B)$ are known. Express the following probabilities in terms of the three known probabilities above.

(a) $P(A \cap B)$

(b) $P(A \cap B^c)$

(c) $P(B \cup (A \cap B^c))$

(d) $P(A^c \cap B^c)$

Solution:

(a) $P(A \cap B) = \boxed{P(A) + P(B) - P(A \cup B)}$. This property is shown in class.

(b) We have seen[3] in class that $P(A \cap B^c) = P(A) - P(A \cap B)$. Plugging in the expression for $P(A \cap B)$ from the previous part, we have

$$P(A \cap B^c) = P(A) - (P(A) + P(B) - P(A \cup B)) = \boxed{P(A \cup B) - P(B)}.$$

Alternatively, we can start from scratch with the set identity $A \cup B = B \cup (A \cap B^c)$ whose union is a disjoint union. Hence,

$$P(A \cup B) = P(B) + P(A \cap B^c).$$

Moving $P(B)$ to the LHS finishes the proof.

(c) $P(B \cup (A \cap B^c)) = \boxed{P(A \cup B)}$ because $A \cup B = B \cup (A \cap B^c)$.

(d) $P(A^c \cap B^c) = \boxed{1 - P(A \cup B)}$ because $A^c \cap B^c = (A \cup B)^c$.

[3] This shows up when we try to derive the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. The key idea is that the set A can be expressed as a disjoint union between $A \cap B$ and $A \cap B^c$. Therefore, by finite additivity, $P(A) = P(A \cap B) + P(A \cap B^c)$. It is easier to visualize this via the Venn diagram.

ECS 315: Probability and Random Processes **2014/1**

HW Solution 3 — Due: Sep 11

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)
The extra question at the end is optional.

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1.

(a) Suppose that $P(A|B) = 0.4$ and $P(B) = 0.5$ Determine the following:

(i) $P(A \cap B)$

(ii) $P(A^c \cap B)$

[Montgomery and Runger, 2010, Q2-105]

(b) Suppose that $P(A|B) = 0.2$, $P(A|B^c) = 0.3$ and $P(B) = 0.8$ What is $P(A)$? [Montgomery and Runger, 2010, Q2-106]

Solution:

(a) Recall that $P(A \cap B) = P(A|B)P(B)$. Therefore,

(i) $P(A \cap B) = 0.4 \times 0.5 = \boxed{0.2.}$

(ii) $P(A^c \cap B) = P(B \setminus A) = P(B) - P(A \cap B) = 0.5 - 0.2 = \boxed{0.3.}$
Alternatively, $P(A^c \cap B) = P(A^c|B)P(B) = (1 - P(A|B))P(B) = (1 - 0.4) \times 0.5 = 0.3$.

(b) By the total probability formula, $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0.2 \times 0.8 + 0.3 \times (1 - 0.8) = \boxed{0.22.}$

Problem 2. Due to an Internet configuration error, packets sent from New York to Los Angeles are routed through El Paso, Texas with probability 3/4. Given that a packet is routed through El Paso, suppose it has conditional probability 1/3 of being dropped. Given that a packet is not routed through El Paso, suppose it has conditional probability 1/4 of being dropped.

(a) Find the probability that a packet is dropped.
Hint: Use total probability theorem.

(b) Find the conditional probability that a packet is routed through El Paso given that it is not dropped.
Hint: Use Bayes' theorem.

[Gubner, 2006, Ex.1.20]

Solution: To solve this problem, we use the notation $E = \{\text{routed through El Paso}\}$ and $D = \{\text{packet is dropped}\}$. With this notation, it is easy to interpret the problem as telling us that

$$P(D|E) = 1/3, \quad P(D|E^c) = 1/4, \quad \text{and } P(E) = 3/4.$$

(a) By the law of total probability,

$$\begin{aligned} P(D) &= P(D|E)P(E) + P(D|E^c)P(E^c) = (1/3)(3/4) + (1/4)(1 - 3/4) \\ &= 1/4 + 1/16 = \boxed{5/16} = 0.3125. \end{aligned}$$

(b) $P(E|D^c) = \frac{P(E \cap D^c)}{P(D^c)} = \frac{P(D^c|E)P(E)}{P(D^c)} = \frac{(1-1/3)(3/4)}{1-5/16} = \boxed{\frac{8}{11}} \approx 0.7273.$

Problem 3. You have two coins, a fair one with probability of heads $\frac{1}{2}$ and an unfair one with probability of heads $\frac{1}{3}$, but otherwise identical. A coin is selected at random and tossed, falling heads up. How likely is it that it is the fair one? [Capinski and Zastawniak, 2003, Q7.28]

Solution: Let F, U, and H be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively.

Because the coin is selected at random, the probability $P(F)$ of selecting the fair coin is $P(F) = \frac{1}{2}$. For fair coin, the conditional probability $P(H|F)$ of heads is $\frac{1}{2}$ For the unfair coin, $P(U) = 1 - P(F) = \frac{1}{2}$ and $P(H|U) = \frac{1}{3}$.

By the Bayes' formula, the probability that the fair coin has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{3}} = \frac{1}{1 + \frac{2}{3}} = \boxed{\frac{3}{5}}.$$

Problem 4. You have three coins in your pocket, two fair ones but the third biased with probability of heads p and tails $1-p$. One coin selected at random drops to the floor, landing heads up. How likely is it that it is one of the fair coins? [Capinski and Zastawniak, 2003, Q7.29]

Solution: Let F, U, and H be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively. We are given that

$$P(F) = \frac{2}{3}, \quad P(U) = \frac{1}{3}, \quad P(H|F) = \frac{1}{2}, P(H|U) = p.$$

By Bayes' theorem, the probability that one of the fair coins has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2} \times \frac{2}{3}}{\frac{1}{2} \times \frac{2}{3} + p \times \frac{1}{3}} = \boxed{\frac{1}{1+p}}.$$

Alternative Solution: Let F_1, F_2, U and H be the events that "the selected coin is the first fair coin", "the selected coin is the second fair coin", "the selected coin is unfair", and "the coin lands heads up", respectively.

Because the coin is selected at random, the events $F1, F2$, and U are equally likely:

$$P(F_1) = P(F_2) = P(U) = \frac{1}{3}.$$

For fair coins, the conditional probability of heads is $\frac{1}{2}$ and for the unfair coin, the conditional probability of heads is p:

$$P(H|F_1) = P(H|F_2) = \frac{1}{2}, \quad P(H|U) = p.$$

The probability that one of the fair coins has been selected given that it lands heads up is $P(F_1 \cup F_2|H)$. Now F_1 and F_2 are disjoint events. Therefore,

$$P(F_1 \cup F_2|H) = P(F_1|H) + P(F_2|H).$$

By Bayes' theorem,

$$P(F_1|H) = \frac{P(H|F_1)P(F_1)}{P(H)} \quad \text{and} \quad P(F_2|H) = \frac{P(H|F_2)P(F_2)}{P(H)}.$$

Therefore,

$$P(F_1 \cup F_2|H) = \frac{P(H|F_1)P(F_1)}{P(H)} + \frac{P(H|F_2)P(F_2)}{P(H)} = \frac{P(H|F_1)P(F_1) + P(H|F_2)P(F_2)}{P(H)}.$$

Note that the collection of three events F_1, F_2, and U partitions the sample space. Therefore, by the total probability theorem,

$$P(H) = P(H|F_1)P(F_1) + P(H|F_2)P(F_2) + P(H|U)P(U).$$

Plugging the above expression of $P(H)$ into our expression for $P(F_1 \cup F_2|H)$ gives

$$\begin{aligned} P(F_1 \cup F_2|H) &= \frac{P(H|F_1)P(F_1) + P(H|F_2)P(F_2)}{P(H|F_1)P(F_1) + P(H|F_2)P(F_2) + P(H|U)P(U)} \\ &= \frac{\frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{3}}{\frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{3} + p\times\frac{1}{3}} = \boxed{\frac{1}{1+p}}. \end{aligned}$$

Problem 5. Someone has rolled a fair dice twice. You know that one of the rolls turned up a face value of six. What is the probability that the other roll turned up a six as well? [Tijms, 2007, Example 8.1, p. 244]

Hint: Not $\frac{1}{6}$.

Solution: Take as sample space the set $\{(i,j)|i,j = 1,\ldots,6\}$, where i and j denote the outcomes of the first and second rolls. A probability of 1/36 is assigned to each element of the sample space. The event of two sixes is given by $A = \{(6,6)\}$ and the event of at least one six is given by $B = (1,6),\ldots,(5,6),(6,6),(6,5),\ldots,(6,1)$. Applying the definition of conditional probability gives

$$P(A|B) = P(A\cap B)/P(B) = \frac{1/36}{11/36}.$$

Hence the desired probability is $\boxed{1/11}$.

Problem 6. Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 99% of the time.

(a) What is $P(-|H)$, the conditional probability that a person tests negative given that the person does have the HIV virus?

(b) What is $P(H|+)$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive?

Solution:

(a) Because the test is correct 99% of the time,

$$P(-|H) = P(+|H^c) = \boxed{0.01}.$$

(b) Using Bayes' formula, $P(H|+) = \frac{P(+|H)P(H)}{P(+)}$, where $P(+)$ can be evaluated by the total probability formula:

$$P(+) = P(+|H)P(H) + P(+|H^c)P(H^c) = 0.99 \times 0.0002 + 0.01 \times 0.9998.$$

Plugging this back into the Bayes' formula gives

$$P(H|+) = \frac{0.99 \times 0.0002}{0.99 \times 0.0002 + 0.01 \times 0.9998} \approx \boxed{0.0194}.$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 2%. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Extra Question

Here are optional questions for those who want more practice.

Problem 7.

(a) Suppose that $P(A|B) = 1/3$ and $P(A|B^c) = 1/4$. Find the range of the possible values for $P(A)$.

(b) Suppose that C_1, C_2, and C_3 partition Ω. Furthermore, suppose we know that $P(A|C_1) = 1/3$, $P(A|C_2) = 1/4$ and $P(A|C_3) = 1/5$. Find the range of the possible values for $P(A)$.

Solution: First recall the total probability theorem: Suppose we have a collection of events $B_1, B_2, \ldots, B_n$ which partitions Ω. Then,

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \cdots P(A \cap B_n) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots P(A|B_n)P(B_n) \end{aligned}$$

(a) Note that B and B^c partition Ω. So, we can apply the total probability theorem:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)).$$

You may check that, by varying the value of $P(B)$ from 0 to 1, we can get the value of $P(A)$ to be any number in the range $\left[\frac{1}{4}, \frac{1}{3}\right]$. Technically, we can not use $P(B) = 0$ because that would make $P(A|B)$ not well-defined. Similarly, we can not use $P(B) = 1$ because that would mean $P(B^c) = 0$ and hence make $P(A|B^c)$ not well-defined. Therfore, the range of $P(A)$ is $\boxed{\left(\frac{1}{4}, \frac{1}{3}\right).}$

Note that larger value of $P(A)$ is not possible because

$$P(A) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)) < \frac{1}{3}P(B) + \frac{1}{3}(1 - P(B)) = \frac{1}{3}.$$

Similarly, smaller value of $P(A)$ is not possible because

$$P(A) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)) > \frac{1}{4}P(B) + \frac{1}{3}(1 - P(B)) = \frac{1}{4}.$$

(b) Again, we apply the total probability theorem:

$$\begin{aligned} P(A) &= P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + P(A|C_3)P(C_3) \\ &= \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3). \end{aligned}$$

Because C_1, C_2, and C_3 partition Ω, we know that $P(C_1) + P(C_2) + P(C_3) = 1$. Now,

$$P(A) = \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3) < \frac{1}{3}P(C_1) + \frac{1}{3}P(C_2) + \frac{1}{3}P(C_3) = \frac{1}{3}.$$

Similarly,

$$P(A) = \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3) > \frac{1}{5}P(C_1) + \frac{1}{5}P(C_2) + \frac{1}{5}P(C_3) = \frac{1}{5}.$$

Therefore, $P(A)$ must be inside $\left(\frac{1}{5}, \frac{1}{3}\right)$.

You may check that any value of $P(A)$ in the range $\boxed{\left(\frac{1}{5}, \frac{1}{3}\right)}$ can be obtained by first setting the value of $P(C_2)$ to be close to 0 and varying the value of $P(C_1)$ from 0 to 1.

Problem 8. Software to detect fraud in consumer phone cards tracks the number of metropolitan areas where calls originate each day. It is found that 1% of the legitimate users originate calls from two or more metropolitan areas in a single day. However, 30% of fraudulent users originate calls from two or more metropolitan areas in a single day. The proportion of fraudulent users is 0.01%. If the same user originates calls from two or more metropolitan areas in a single day, what is the probability that the user is fraudulent? [Montgomery and Runger, 2010, Q2-144]

Solution: Let F denote the event of fraudulent user and let M denote the event of originating calls from multiple (two or more) metropolitan areas in a day. Then,

$$\begin{aligned}
P\left(F\left|M\right.\right) &= \frac{P\left(M\left|F\right.\right)P\left(F\right)}{P\left(M\left|F\right.\right)P\left(F\right)+P\left(M|F^c\right)P\left(F^c\right)} = \frac{1}{1+\frac{P(M|F^c)}{P(M|F)}\times\frac{P(F^c)}{P(F)}} \\
&= \frac{1}{1+\frac{\frac{1}{100}}{\frac{30}{100}}\times\frac{\frac{9999}{10^4}}{\frac{1}{10^4}}} = \frac{1}{1+\frac{9999}{30}} = \frac{30}{30+9999} = \frac{30}{10029} \approx \boxed{0.0030}.
\end{aligned}$$

Problem 9. In his book *Chances: Risk and Odds in Everyday Life*, James Burke says that there is a 72% chance a polygraph test (lie detector test) will catch a person who is, in fact, lying. Furthermore, there is approximately a 7% chance that the polygraph will falsely accuse someone of lying.

(a) If the polygraph indicated that 30% of the questions were answered with lies, what would you estimate for the actual percentage of lies in the answers?

(b) If the polygraph indicated that 70% of the questions were answered with lies, what would you estimate for the actual percentage of lies?

[Brase and Brase, 2011, Q4.2.26]

Solution: Let AT and AL be the events that "the person actually answers the truth" and "the person actually answers with lie", respectively. Also, let PT and PL be the events that "the polygraph indicates that the answer is the truth" and "the polygraph indicates that the answer is a lie", respectively.

We know, from the provided information, that $P[PL|AL] = 0.72$ and that $P[PL|AT] = 0.07$.

Applying the total probability theorem, we have

$$\begin{aligned}
P(PL) &= P(PL|AL)P(AL) + P(PL|AT)P(AT) \\
&= P(PL|AL)P(AL) + P(PL|AT)(1-P(AL)).
\end{aligned}$$

Solving for $P(AL)$, we have

$$P(AL) = \frac{P(PL) - P(PL|AT)}{P(PL|AL) - P(PL|AT)} = \frac{P(PL) - 0.07}{0.72 - 0.07} = \frac{P(PL) - 0.07}{0.65}.$$

(a) Plugging in $P(PL) = 0.3$, we have $P(AL) = \boxed{0.3538}$.

(b) Plugging in $P(PL) = 0.7$, we have $P(AL) = \boxed{0.9692}$.

Problem 10. An article in the British Medical Journal ["Comparison of Treatment of Renal Calculi by Operative Surgery, Percutaneous Nephrolithotomy, and Extracorporeal Shock Wave Lithotripsy" (1986, Vol. 82, pp. 879892)] provided the following discussion of success rates in kidney stone removals. Open surgery (OS) had a success rate of 78% (273/350) while a newer method, percutaneous nephrolithotomy (PN), had a success rate of 83% (289/350). This newer method looked better, but the results changed when stone diameter was considered. For stones with diameters less than two centimeters, 93% (81/87) of cases of open surgery were successful compared with only 87% (234/270) of cases of PN. For stones greater than or equal to two centimeters, the success rates were 73% (192/263) and 69% (55/80) for open surgery and PN, respectively. Open surgery is better for both stone sizes, but less successful in total. In 1951, E. H. Simpson pointed out this apparent contradiction (known as Simpson's Paradox) but the hazard still persists today. Explain how open surgery can be better for both stone sizes but worse in total. [Montgomery and Runger, 2010, Q2-115]

Solution: First, let's recall the total probability theorem:

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A|B)\,P(B) + P(A|B^c)\,P(B^c). \end{aligned}$$

We can see that $P(A)$ does not depend only on $P(A \cap B)$ and $P(A|B^c)$. It also depends on $P(B)$ and $P(B^c)$. In the extreme case, we may imagine the case with $P(B) = 1$ in which $P(A) = P(A|B)$. At another extreme, we may imagine the case with $P(B) = 0$ in which $P(A) = P(A|B^c)$. Therefore, depending on the value of $P(B)$, the value of $P(A)$ can be anywhere between $P(A|B)$ and $P(A|B^c)$.

Now, let's consider events A_1, B_1, A_2, and B_2. Let $P(A_1|B_1) = 0.93$ and $P(A_1|B_1^c) = 0.73$. Therefore, $P(A_1) \in [0.73, 0.93]$. On the other hand, let $P(A_2|B_2) = 0.87$ and $P(A_2|B_2^c) = 0.69$. Therefore, $P(A_2) \in [0.69, 0.87]$. With small value of $P(B_1)$, the value of $P(A_1)$ can be 0.78 which is closer to its lower end of the bound. With large value of $P(B_2)$, the value of $P(A_2)$ can be 0.83 which is closer to its upper end of the bound. Therefore, even though $P(A_1|B_1) > P(A_2|B_2) = 0.87$ and $P(A_1|B_1^c) > P(A_2|B_2^c)$, it is possible that $P(A_1) < P(A_2)$.

In the context of the paradox under consideration, note that the success rate of PN with small stones (87%) is higher than the success rate of OS with large stones (73%). Therefore, by having a lot of large stone cases to be tested under OS and also have a lot of small stone cases to be tested under PN, we can create a situation where the overall success rate of PN comes out to be better then the success rate of OS. This is exactly what happened in the study as shown in Table 3.1.

Open surgery

	success	failure	sample size	sample percentage	conditional success rate
large stone	192	71	263	75%	73%
small stone	81	6	87	25%	93%
overall summary	273	77	350	100%	78%

PN

	success	failure	sample size	sample percentage	conditional success rate
large stone	55	25	80	23%	69%
small stone	234	36	270	77%	87%
overall summary	289	61	350	100%	83%

Table 3.1: Success rates in kidney stone removals.

ECS 315: Probability and Random Processes **2014/1**

HW Solution 4 — Due: September 18

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)
The extra questions at the end are optional.

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases. A specific design is randomly generated by the Web server when you visit the site. Let A denote the event that the design color is red and let B denote the event that the font size is not the smallest one.

(a) Use classical probability to evaluate $P(A)$, $P(B)$ and $P(A \cap B)$. Show that the two events A and B are independent by checking whether $P(A \cap B) = P(A)P(B)$.

(b) Using the values of $P(A)$ and $P(B)$ from the previous part and the fact that $A \perp\!\!\!\perp B$, calculate the following probabilities.

(i) $P(A \cup B)$

(ii) $P(A \cup B^c)$

(iii) $P(A^c \cup B^c)$

[Montgomery and Runger, 2010, Q2-84]

Solution:

(a) By multiplication rule, there are

$$|\Omega| = 4 \times 3 \times 5 \times 3 \times 5 \tag{4.1}$$

possible designs. The number of designs whose color is red is given by

$$|A| = 1 \times 3 \times 5 \times 3 \times 5.$$

Note that the "4" in (4.1) is replaced by "1" because we only consider one color (red). Therefore,

$$P(A) = \frac{1 \times 3 \times 5 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{1}{4}.}$$

Similarly, $|B| = 4 \times 3 \times 4 \times 3 \times 5$ where the "5" in the middle of (4.1) is replaced by "4" because we can't use the smallest font size. Therefore,

$$P(B) = \frac{4 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{4}{5}.}$$

For the event $A \cap B$, we replace "4" in (4.1) by "1" because we need red color and we replace "5" in the middle of (4.1) by "4" because we can't use the smallest font size. This gives

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{1 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \frac{1 \times 4}{4 \times 5} = \boxed{\frac{1}{5}} = 0.2.$$

Because $P(A \cap B) = P(A)P(B)$, the events A and B are independent.

(b)

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{4}{5} - \frac{1}{5} = \boxed{\frac{17}{20} = 0.85.}$

(ii) **Method 1**: $P(A \cup B^c) = 1 - P\left((A \cup B^c)^c\right) = 1 - P(A^c \cap B)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$. Hence, $P(A^{\cancel{c}} \cup B^c) = 1 - P(A^c)P(B) = 1 - \frac{3}{4}\frac{4}{5} = \frac{2}{5} = \boxed{0.4.}$

Method 2: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = B^c \cup (A \cap B)$. Therefore,

$$P(A \cup B^c) = P(B^c) + P(A \cap B) = 1 - P(B) + P(A)P(B) = 1 - \frac{4}{5} + \frac{1}{4}\frac{4}{5} = \frac{2}{5}.$$

Method 3: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = A \cup (A^c \cap B^c)$. Therefore, $P(A \cup B^c) = P(A) + P(A^c \cap B^c)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B^c$. Hence,

$$P(A \cup B^c) = P(A) + P(A^c)P(B^c) = P(A) + (1 - P(A))(1 - P(B)) = \frac{1}{4} + \frac{3}{4}\frac{1}{5} = \frac{2}{5}.$$

(iii) **Method 1**: $P(A^c \cup B^c) = 1 - P\left((A^c \cup B^c)^c\right) = 1 - P(A \cap B) = 1 - 0.2 = \boxed{0.8.}$
Method 2: From the Venn diagram, note that $A^c \cup B^c$ can be expressed as a disjoint union: $A^c \cup B^c = (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c)$. Therefore,

$$P\left(A^c \cup B^c\right) = P\left(A^c \cap B\right) + P\left(A \cap B^c\right) + P\left(A^c \cap B^c\right).$$

Now, because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$, $A \perp\!\!\!\perp B^c$, and $A^c \perp\!\!\!\perp B^c$. Hence,

$$\begin{aligned} P\left(A^c \cup B^c\right) &= P\left(A^c\right) P\left(B\right) + P\left(A\right) P\left(B^c\right) + P\left(A^c\right) P\left(B^c\right) \\ &= \left(1 - P\left(A\right)\right) P\left(B\right) + P\left(A\right)\left(1 - P\left(B\right)\right) + \left(1 - P\left(A\right)\right)\left(1 - P\left(B\right)\right) \\ &= \frac{3}{4} \times \frac{4}{5} + \frac{1}{4} \times \frac{1}{5} + \frac{3}{4} \times \frac{1}{5} = \frac{16}{20} = \frac{4}{5} \end{aligned}$$

Problem 2. In this question, each experiment has equiprobable outcomes.

(a) Let $\Omega = \{1, 2, 3, 4\}$, $A_1 = \{1, 2\}$, $A_2 = \{1, 3\}$, $A_3 = \{2, 3\}$.

(i) Determine whether $P\left(A_i \cap A_j\right) = P\left(A_i\right) P\left(A_j\right)$ for all $i \neq j$.

(ii) Check whether $P\left(A_1 \cap A_2 \cap A_3\right) = P\left(A_1\right) P\left(A_2\right) P\left(A_3\right)$.

(iii) Are A_1, A_2, and A_3 independent?

(b) Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2, 3, 4\}$, $A_2 = A_3 = \{4, 5, 6\}$.

(i) Check whether $P\left(A_1 \cap A_2 \cap A_3\right) = P\left(A_1\right) P\left(A_2\right) P\left(A_3\right)$.

(ii) Check whether $P\left(A_i \cap A_j\right) = P\left(A_i\right) P\left(A_j\right)$ for all $i \neq j$.

(iii) Are A_1, A_2, and A_3 independent?

Solution:

(a) We have $P(A_i) = \frac{1}{2}$ and $P(A_i \cap A_j) = \frac{1}{4}$.

(i) $P(A_i \cap A_j) = P(A_i)P(A_j)$ for any $i \neq j$.

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$. Hence, $P\left(A_1 \cap A_2 \cap A_3\right) = 0$, which is *not* the same as $P\left(A_1\right) P\left(A_2\right) P\left(A_3\right)$.

(iii) No.

Remark: This counter-example shows that pairwise independence does not imply independence.

(b) We have $P(A_1) = \frac{4}{6} = \frac{2}{3}$ and $P(A_2) = P(A_3) = \frac{3}{6} = \frac{1}{2}$.

(i) $A_1 \cap A_2 \cap A_3 = \{4\}$. Hence, $P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}$.
$P(A_1) P(A_2) P(A_3) = \frac{2}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{6}$.
Hence, $P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$.

(ii) $P(A_2 \cap A_3) = P(A_2) = \frac{1}{2} \neq P(A_2)P(A_3)$
$P(A_1 \cap A_2) = p(4) = \frac{1}{6} \neq P(A_1)P(A_2)$
$P(A_1 \cap A_3) = p(4) = \frac{1}{6} \neq P(A_1)P(A_3)$
Hence, $P(A_i \cap A_j) \neq P(A_i) P(A_j)$ for all $i \neq j$.

(iii) No.

Remark: This counter-example shows that one product condition does not imply independence.

Problem 3. In an experiment, A, B, C, and D are events with probabilities $P(A \cup B) = \frac{5}{8}$, $P(A) = \frac{3}{8}$, $P(C \cap D) = \frac{1}{3}$, and $P(C) = \frac{1}{2}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find

(i) $P(A \cap B)$

(ii) $P(B)$

(iii) $P(A \cap B^c)$

(iv) $P(A \cup B^c)$

(b) Are A and B independent?

(c) Find

(i) $P(D)$

(ii) $P(C \cap D^c)$

(iii) $P(C^c \cap D^c)$

(iv) $P(C|D)$

(v) $P(C \cup D)$

(vi) $P(C \cup D^c)$

(d) Are C and D^c independent?

Solution:

(a)

(i) Because $A \perp B$, we have $A \cap B = \emptyset$ and hence $P(A \cap B) = \boxed{0}$.

(ii) Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, $P(B) = P(A \cup B) - P(A) + P(A \cap B) = 5/8 - 3/8 + 0 = 2/8 = \boxed{1/4}$.

(iii) $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) = \boxed{3/8}$.

(iv) Start with $P(A \cup B^c) = 1 - P(A^c \cap B)$. Now, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) = 1/4$. Hence, $P(A \cup B^c) = 1 - 1/4 = \boxed{3/4}$.

(b) Events A and B are not independent because $P(A \cap B) \neq P(A)P(B)$.

(c)

(i) Because $C \perp\!\!\!\perp D$, we have $P(C \cap D) = P(C)P(D)$. Hence, $P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1/3}{1/2} = \boxed{2/3}$.

(ii) **Method 1**: $P(C \cap D^c) = P(C) - P(C \cap D) = 1/2 - 1/3 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C \perp\!\!\!\perp D^c$. Hence, $P(C \cap D^c) = P(C)P(D^c) = \frac{1}{2}\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iii) **Method 1**: First, we find $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = 5/6$. Hence, $P(C^c \cap D^c) = 1 - P(C \cup D) = 1 - 5/6 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C^c \perp\!\!\!\perp D^c$. Hence, $P(C^c \cap D^c) = P(C^c)P(D^c) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iv) Because $C \perp\!\!\!\perp D$, we have $P(C|D) = P(C) = \boxed{1/2}$.

(v) In part (iii), we already found $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = \boxed{5/6}$.

(vi) **Method 1**: $P(C \cup D^c) = 1 - P(C^c \cap D) = 1 - P(C^c)P(D) = 1 - \frac{1}{2}\frac{2}{3} = \boxed{2/3}$. Note that we use the fact that $C^c \perp\!\!\!\perp D$ to get the second equality.
Method 2: Alternatively, $P(C \cup D^c) = P(C) + P(D^c) - P(C \cap D^C)$. From (i), we have $P(D) = 2/3$. Hence, $P(D^c) = 1 - 2/3 = 1/3$. From (ii), we have $P(C \cap D^C) = 1/6$. Therefore, $P(C \cup D^c) = 1/2 + 1/3 - 1/6 = 2/3$.

(d) Yes. We know that if $C \perp\!\!\!\perp D$, then $C \perp\!\!\!\perp D^c$.

Figure 4.1: Circuit for Problem 4

Problem 4. Series Circuit: The circuit in Figure 4.1 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-32]

Solution: Let L and R denote the events that the left and right devices operate, respectively. For a path to exist, both need to operate. Therefore, the probability that the circuit operates is $P(L \cap R)$.

We are told that $L^c \perp\!\!\!\perp R^c$. This is equivalent to $L \perp\!\!\!\perp R$. By their independence,

$$P(L \cap R) = P(L)P(R) = 0.8 \times 0.9 = \boxed{0.72}.$$

Extra Questions

Here are optional questions for those who want more practice.

Problem 5. Show that if A and B are independent events, then so are A and B^c, A^c and B, and A^c and B^c.

Solution: To show that two events C_1 and C_2 are independent, we need to show that $P(C_1 \cap C_2) = P(C_1)P(C_2)$.

(a) Note that
$$P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B).$$
Because $A \perp\!\!\!\perp B$, the last term can be factored in to $P(A)P(B)$ and hence
$$P(A \cap B^c) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$$

(b) By interchanging the role of A and B in the previous part, we have
$$P(A^c \cap B) = P(B \cap A^c) = P(B)\,P(A^c).$$

(c) From set theory, we know that $A^c \cap B^c = (A \cup B)^c$. Therefore,

$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B),$$

where, for the last equality, we use

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

which is discussed in class.

Because $A \perp\!\!\!\perp B$, we have

$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c). \end{aligned}$$

Remark: By interchanging the roles of A and A^c and/or B and B^c, it follows that if any one of the four pairs is independent, then so are the other three. [Gubner, 2006, p.31]

Problem 6. Anne and Betty go fishing. Find the conditional probability that Anne catches no fish given that at least one of them catches no fish. Assume they catch fish independently and that each has probability $0 < p < 1$ of catching no fish. [Gubner, 2006, Q2.62]

Hint: Let A be the event that Anne catches no fish and B be the event that Betty catches no fish. Observe that the question asks you to evaluate $P(A|(A \cup B))$.

Solution: From the question, we know that A and B are independent. The event "at least one of the two women catches nothing" can be represented by $A \cup B$. So we have

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B) - P(A)P(B)} = \frac{p}{2p - p^2} = \boxed{\frac{1}{2-p}}.$$

Problem 7. The circuit in Figure 4.2 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-34]

Solution: Let T and B denote the events that the top and bottom devices operate, respectively. There is a path if at least one device operates. Therefore, the probability that the circuit operates is $P(T \cup B)$. Note that

$$P(T \cup B) = 1 - P((T \cup B)^c) = 1 - P(T^c \cap B^c).$$

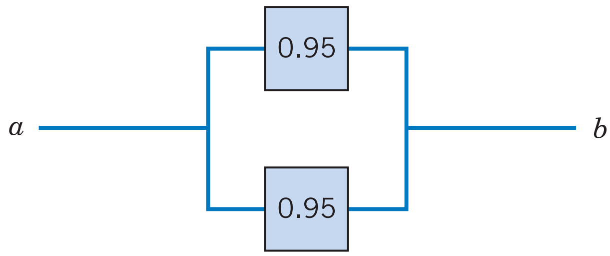


Figure 4.2: Circuit for Problem 7

We are told that $T^c \perp\!\!\!\perp B^c$. By their independence,

$$P(T^c \cap B^c) = P(T^c)P(B^c) = (1-0.95)\times(1-0.95) = 0.05^2 = 0.0025.$$

Therefore,

$$P(T \cup B) = 1 - P(T^c \cap B^c) = 1 - 0.0025 = \boxed{0.9975}.$$

Problem 8. The circuit in Figure 4.3 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-35]

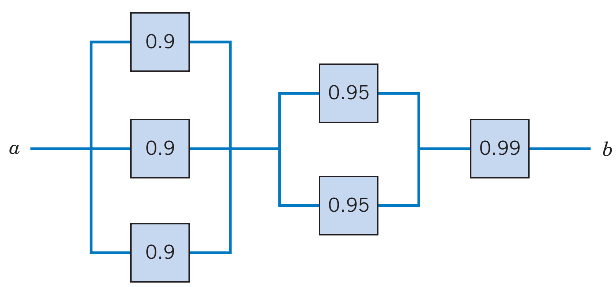


Figure 4.3: Circuit for Problem 8

Solution: The solution can be obtained from a partition of the graph into three columns. Let L denote the event that there is a path of functional devices only through the three units on the left. From the independence and based upon Problem 7,

$$P(L) = 1 - (1-0.9)^3 = 1 - 0.1^3 = 0.999.$$

Similarly, let M denote the event that there is a path of functional devices only through the two units in the middle. Then,

$$P(M) = 1 - (1 - 0.95)^2 = 1 - 0.05^2 = 1 - 0.0025 = 0.9975.$$

Finally, the probability that there is a path of functional devices only through the one unit on the right is simply the probability that the device functions, namely, 0.99.

Therefore, with the independence assumption used again, along with similar reasoning to the solution of Problem 4, the solution is

$$0.999 \times 0.9975 \times 0.99 = 0.986537475 \approx \boxed{0.987}.$$

Problem 9. Show that

(a) $P(A \cap B \cap C) = P(A) \times P(B|A) \times P(C|A \cap B)$.

(b) $P(B \cap C|A) = P(B|A)P(C|B \cap A)$

Solution:

(a) We can see directly from the definition of $P(B|A)$ that

$$P(A \cap B) = P(A)P(B|A).$$

Similarly, when we consider event $A \cap B$ and event C, we have

$$P(A \cap B \cap C) = P(A \cap B)\, P(C|A \cap B)\,.$$

Combining the two equalities above, we have

$$P(A \cap B \cap C) = P(A) \times P(B|A) \times P(C|A \cap B)\,.$$

(b) By definition,

$$P(B \cap C|A) = \frac{P(A \cap B \cap C)}{P(A)}.$$

Substitute $P(A \cap B \cap C)$ from part (a) to get

$$P(B \cap C|A) = \frac{P(A) \times P(B|A) \times P(C|A \cap B)}{P(A)} = P(B|A) \times P(C|A \cap B)\,.$$

ECS 315: Probability and Random Processes **2014/1**

HW Solution 5 — Due: Sep 25

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)
The extra question at the end is optional.

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1 (Majority Voting in Digital Communication)**.** A certain binary communication system has a bit-error rate of 0.1; i.e., in transmitting a single bit, the probability of receiving the bit in error is 0.1. To transmit messages, a three-bit repetition code is used. In other words, to send the message 1, a "codeword" 111 is transmitted, and to send the message 0, a "codeword" 000 is transmitted. At the receiver, if two or more 1s are received, the decoder decides that message 1 was sent; otherwise, i.e., if two or more zeros are received, it decides that message 0 was sent.

Assuming bit errors occur independently, find the probability that the decoder puts out the wrong message.

[Gubner, 2006, Q2.62]

Solution: Let $p = 0.1$ be the bit error rate. Let $\mathcal{E}$ be the error event. (This is the event that the decoded bit value is not the same as the transmitted bit value.) Because majority voting is used, event $\mathcal{E}$ occurs if and only if there are at least two bit errors. Therefore

$$P(\mathcal{E}) = \binom{3}{2}p^2(1-p) + \binom{3}{3}p^3 = p^2(3-2p).$$

When $p = 0.1$, we have $P(\mathcal{E}) \approx \boxed{0.028}$.

Problem 2. An optical inspection system is to distinguish among different part types. The probability of a correct classification of any part is 0.98. Suppose that three parts are inspected and that the classifications are independent.

(a) Let the random variable X denote the number of parts that are correctly classified. Determine the probability mass function of X. [Montgomery and Runger, 2010, Q3-20]

(b) Let the random variable Y denote the number of parts that are incorrectly classified. Determine the probability mass function of Y.

Solution[1]:

We will reexpress the problem in terms of Bernoulli trials so that we can use the results discussed in class. In this problem, we have three Bernoulli trials. Each trial deals with classification.

(a) To find $p_X(x)$, first we find its support. Three parts are inspected here. Therefore, X can be 0, 1, 2, or 3. So, we need to find $p_X(x) = P[X = x]$ when $x = 0, 1, 2$ or 3. The pmf $p_X(x)$ for other x values are all 0 because X cannot take the value of those x.

For each $x \in \{0, 1, 2, 3\}$, $p_X(x) = P[X = x]$ is simply the probability that exactly x parts are correctly classified. Note that, because we are interested in the correctly classified part, we define the "success" event for a trial to be the event that the part is classified correctly. We are given that the probability of a correct classification of any part is 0.98. Therefore, for each of our Bernoulli trials, the probability of success is $p = 0.98$. Under such interpretation (of "success"), $p_X(x)$ is then the same as finding the probability of having exactly x successes in $n = 3$ Bernoulli trials. We have seen in class that the probability of this is $\binom{3}{x}p^x(1-p)^{3-x}$. Plugging in $p = 0.98$, we have $p_X(x) = \binom{3}{x}0.98^x(0.02)^{3-x}$ for $x \in \{0, 1, 2, 3\}$.

Combining the expression above with the cases for other x values, we then have

$$p_X(x) = \begin{cases} \binom{3}{x}0.98^x(0.02)^{3-x}, & x \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

In particular, $p_X(0) = 8 \times 10^{-6}$, $p_X(1) = 0.001176$, $p_X(2) = 0.057624$, and $p_X(3) = 0.941192$. In fact, this X is a binomial random variable with $n = 3$ and $p = 0.98$. In `MATLAB`, the probabilities above can be calculated via the command
`binopdf(0:3,3,0.98)`.

[1] The solution provided here assumes that we still haven't reached the part of the course where binomial random variable is discussed. Therefore, the pmf is derived by relying on the concept of Bernoulli trials and the formula discussed back when we studied that topic.

(b) ***Method 1***: Similar analysis is performed on the random variable Y. The only difference here is that, now, we are interested in the number of parts that are *incorrectly* classified. Therefore, we will define the "success" event for a trial to be the event that the part is classified *incorrectly*. We are given that the probability of a correct classification of any part is 0.98. Therefore, for each of our Bernoulli trials, the probability of success is $1 - p = 1 - 0.98 = 0.02$. With this new probability of success, we have

$$p_Y(y) = \begin{cases} \binom{3}{y} 0.02^y (0.98)^{3-y}, & y \in \{0,1,2,3\}, \\ 0, & \text{otherwise} \end{cases} \tag{5.2}$$

In particular, $p_Y(0) = 0.941192$, $p_Y(1) = 0.057624$, $p_Y(2) = 0.001176$, and $p_Y(3) = 8 \times 10^{-6}$. In fact, this Y is a binomial random variable with $n = 3$ and $p = 0.02$. In `MATLAB`, the probability values above can be calculated via the command
`binopdf(0:3,3,0.02)`.

Method 2: Alternatively, note that there are three parts. If X of them are classified correctly, then the number of incorrectly classified parts is $n - X$, which is what we defined as Y. Therefore, $Y = 3 - X$. Hence, $p_Y(y) = P[Y = y] = P[3 - X = y] = P[X = 3 - y] = p_X(3 - y)$.

Problem 3. Consider the sample space $\Omega = \{-2,-1,0,1,2,3,4\}$. Suppose that $P(A) = |A|/|\Omega|$ for any event $A \subset \Omega$. Define the random variable $X(\omega) = \omega^2$. Find the probability mass function of X.

Solution: The random variable maps the outcomes $\omega = -2,-1,0,1,2,3,4$ to numbers $x = 4,1,0,1,4,9,16$, respectively. Therefore,

$$\begin{aligned} p_X(0) &= P(\{\omega : X(\omega) = 0\}) = P(\{0\}) = \frac{1}{7}, \\ p_X(1) &= P(\{\omega : X(\omega) = 1\}) = P(\{-1,1\}) = \frac{2}{7}, \\ p_X(4) &= P(\{\omega : X(\omega) = 4\}) = P(\{-2,2\}) = \frac{2}{7}, \\ p_X(9) &= P(\{\omega : X(\omega) = 9\}) = P(\{3\}) = \frac{1}{7}, \text{ and} \\ p_X(16) &= P(\{\omega : X(\omega) = 16\}) = P(\{4\}) = \frac{1}{7}. \end{aligned}$$

Combining the results above, we get the complete pmf:

$$p_X(x) = \boxed{\begin{cases} \frac{1}{7}, & x = 0,9,16, \\ \frac{2}{7}, & x = 1,4, \\ 0, & \text{otherwise.} \end{cases}}$$

Problem 4. Suppose X is a random variable whose pmf at $x = 0, 1, 2, 3, 4$ is given by $p_X(x) = \frac{2x+1}{25}$.

Remark: Note that the statement above does not specify the value of the $p_X(x)$ at the value of x that is not 0,1,2,3, or 4.

(a) What is $p_X(5)$?

(b) Determine the following probabilities:

(i) $P[X = 4]$

(ii) $P[X \le 1]$

(iii) $P[2 \le X < 4]$

(iv) $P[X > -10]$

Solution:

(a) First, we calculate

$$\sum_{x=0}^{4} p_X(x) = \sum_{x=0}^{4} \frac{2x+1}{25} = \frac{25}{25} = 1.$$

Therefore, there can't be any other x with $p_X(x) > 0$. At $x = 5$, we then conclude that $p_X(5) = \boxed{0.}$ The same reasoning also implies that $p_X(x) = 0$ at any x that is not 0,1,2,3, or 4.

(b) Recall that, for discrete random variable X, the probability

$$P[\text{some condition(s) on } X]$$

can be calculated by adding $p_X(x)$ for all x in the support of X that satisfies the given condition(s).

(i) $P[X = 4] = p_X(4) = \frac{2\times4+1}{25} = \boxed{\frac{9}{25}.}$

(ii) $P[X \le 1] = p_X(0) + p_X(1) = \frac{2\times0+1}{25} + \frac{2\times1+1}{25} = \frac{1}{25} + \frac{3}{25} = \boxed{\frac{4}{25}.}$

(iii) $P[2 \le X < 4] = p_X(2) + p_X(3) = \frac{2\times2+1}{25} + \frac{2\times3+1}{25} = \frac{5}{25} + \frac{7}{25} = \boxed{\frac{12}{25}.}$

(iv) $P[X > -10] = \boxed{1}$ because all the x in the support of X satisfies $x > -10$.

Problem 5. The random variable V has pmf

$$p_V(v) = \begin{cases} cv^2, & v = 1,2,3,4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c.

(b) Find $P[V \in \{u^2 : u = 1,2,3,\ldots\}]$.

(c) Find the probability that V is an even number.

(d) Find $P[V > 2]$.

(e) Sketch $p_V(v)$.

(f) Sketch $F_V(v)$. (Note that $F_V(v) = P[V \le v]$.)

Solution: [Y&G, Q2.2.3]

(a) We choose c so that the pmf sums to one:

$$\sum_v p_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1.$$

Hence, $c = \boxed{1/30}$.

(b) $P[V \in \{u^2 : u = 1,2,3,\ldots\}] = p_V(1) + p_V(4) = c(1^2 + 4^2) = \boxed{17/30}$.

(c) $P[V \text{ even}] = p_V(2) + p_V(4) = c(2^2 + 4^2) = 20/30 = \boxed{2/3}$.

(d) $P[V > 2] = p_V(3) + p_V(4) = c(3^2 + 4^2) = 25/30 = \boxed{5/6}$.

(e) See Figure 5.1 for the sketch of $p_V(v)$:

(f) See Figure 5.2 for the sketch of $F_V(v)$:

Extra Question

Problem 6. Consider a transmission over a binary symmetric channel (BSC) with crossover probability p. The random (binary) input to the BSC is denoted by X. Let p_1 be the probability that $X = 1$. (This implies the probability that $X = 0$ is $1 - p_1$.) Let Y by the output of the BSC.

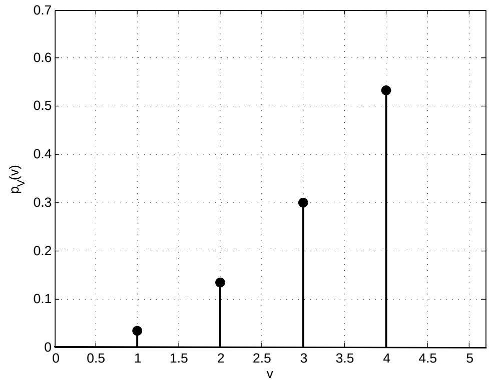


Figure 5.1: Sketch of $p_V(v)$ for Question 5

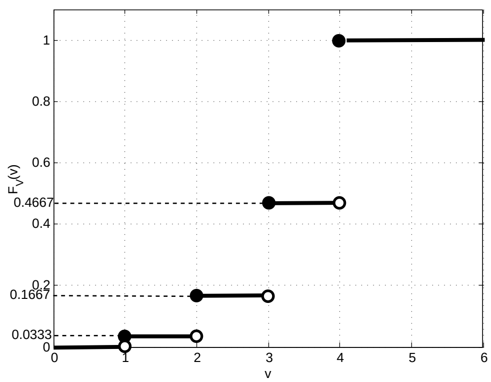


Figure 5.2: Sketch of $F_V(v)$ for Question 5

(a) Suppose, at the receiver (which observes the output of the BSC), we learned that $Y = 1$. For each of the following scenarios, which event is more likely, "$X = 1$ was transmitted" or "$X = 0$ was transmitted"? (Hint: Use Bayes' theorem.)

(i) Assume $p = 0.3$ and $p_1 = 0.1$.

(ii) Assume $p = 0.3$ and $p_1 = 0.5$.

(iii) Assume $p = 0.3$ and $p_1 = 0.9$.

(iv) Assume $p = 0.7$ and $p_1 = 0.5$.

(b) Suppose, at the receiver (which observes the output of the BSC), we learned that $Y = 0$. For each of the following scenarios, which event is more likely, "$X = 1$ was transmitted" or "$X = 0$ was transmitted"?

(i) Assume $p = 0.3$ and $p_1 = 0.1$

(ii) Assume $p = 0.3$ and $p_1 = 0.5$

(iii) Assume $p = 0.3$ and $p_1 = 0.9$

(iv) Assume $p = 0.7$ and $p_1 = 0.5$

Remark: A MAP (maximum a posteriori) detector is a detector that takes the observed value Y and then calculate the most likely transmitted value. More specifically,

$$\hat{x}_{MAP}(y) = \arg\max_x P[X = x|Y = y]$$

In fact, in part (a), each of your answers is $\hat{x}_{MAP}(1)$ and in part (b), each of your answers is $\hat{x}_{MAP}(0)$.

Solution: First, recall that, in class, we define $P[X = x]$ to be $P([X = x])$. Here, we extend the such definition to conditional probability. In particular,

$$P[Y = y\,|X = x] = P([Y = y]\,|[X = x]).$$

Here, we are given that $P[X = 1] = p_1$. Applying $P(A^c) = 1-P(A)$, we have $P[X = 0] = 1 - p_1$. We are also given that

$$P[Y = 1\,|X = 0] = P[Y = 0\,|X = 1] = p.$$

Applying $P(A^c\,|B) = 1 - P(A\,|B)$, we have

$$P[Y = 0\,|X = 0] = P[Y = 1\,|X = 1] = 1 - p.$$

(a) Here, we know that $Y = 1$. To find out what was transmitted, we compare $P[X = 0\,|Y = 1]$ and $P[X = 1\,|Y = 1]$. By Bayes' theorem,

$$P[X = 0\,|Y = 1] = \frac{P[Y = 1\,|X = 0]\,P[X = 0]}{P[Y = 1]} = \frac{p\,(1 - p_1)}{P[Y = 1]} = \frac{p - pp_1}{P[Y = 1]}$$

and

$$P[X = 1\,|Y = 1] = \frac{P[Y = 1\,|X = 1]\,P[X = 1]}{P[Y = 1]} = \frac{(1 - p)\,p_1}{P[Y = 1]} = \frac{p_1 - pp_1}{P[Y = 1]}$$

Note that both terms have "$-pp_1$" in the numerator and "$P[Y = 1]$" the denominator. So, we can simply compare the "p" and "p_1" parts.

(i) When $p = 0.3$ and $p_1 = 0.1$, we have $p > p_1$. Therefore, $P[X = 0\,|Y = 1] > P[X = 1\,|Y = 1]$. In other words, conditioned on $Y = 1$, the event $X = \boxed{0}$ is more likely.

(ii) When $p = 0.3$ and $p_1 = 0.5$, we have $p < p_1$. Therefore, $P[X = 0 | Y = 1] < P[X = 1 | Y = 1]$. In other words, conditioned on $Y = 1$, the event $X = \boxed{1}$ is more likely.

(iii) When $p = 0.3$ and $p_1 = 0.9$, we have $p < p_1$. Therefore, $P[X = 0 | Y = 1] < P[X = 1 | Y = 1]$. In other words, conditioned on $Y = 1$, the event $X = \boxed{1}$ is more likely.

(iv) When $p = 0.7$ and $p_1 = 0.5$, we have $p > p_1$. Therefore, $P[X = 0 | Y = 1] > P[X = 1 | Y = 1]$. In other words, conditioned on $Y = 1$, the event $X = \boxed{0}$ is more likely.

(b) In this part, we know that $Y = 0$. To find out what was transmitted, we compare $P[X = 0 | Y = 0]$ and $P[X = 1 | Y = 0]$. By Bayes' theorem,

$$P[X = 0 | Y = 0] = \frac{P[Y = 0 | X = 0]\, P[X = 0]}{P[Y = 0]} = \frac{(1-p)(1-p_1)}{P[Y = 0]} = \frac{1 - p - p_1 + pp_1}{P[Y = 0]}$$

and

$$P[X = 1 | Y = 0] = \frac{P[Y = 0 | X = 1]\, P[X = 1]}{P[Y = 0]} = \frac{pp_1}{P[Y = 0]} = \frac{pp_1}{P[Y = 0]}$$

Note that both terms have "$-pp_1$" in the numerator and "$P[Y = 0]$" the denominator. So, we can simply compare the "$1 - p - p_1$" and "$''0$ parts.

(i) When $p = 0.3$ and $p_1 = 0.1$, we have $1 - p - p_1 = 0.6 > 0$. Therefore, $P[X = 0 | Y = 0] > P[X = 1 | Y = 0]$. In other words, conditioned on $Y = 0$, the event $X = \boxed{0}$ is more likely.

(ii) When $p = 0.3$ and $p_1 = 0.5$, we have $1 - p - p_1 = 0.2 > 0$. Therefore, $P[X = 0 | Y = 0] > P[X = 1 | Y = 0]$. In other words, conditioned on $Y = 0$, the event $X = \boxed{0}$ is more likely.

(iii) When $p = 0.3$ and $p_1 = 0.9$, we have $1 - p - p_1 = -0.2 < 0$. Therefore, $P[X = 0 | Y = 0] > P[X = 1 | Y = 0]$. In other words, conditioned on $Y = 1$, the event $X = \boxed{1}$ is more likely.

(iv) When $p = 0.7$ and $p_1 = 0.5$,we have $1 - p - p_1 = -0.2 < 0$. Therefore, $P[X = 0 | Y = 0] > P[X = 1 | Y = 0]$. In other words, conditioned on $Y = 0$, the event $X = \boxed{1}$ is more likely.

ECS 315: Probability and Random Processes	2014/1

HW Solution 6 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. The thickness of the wood paneling (in inches) that a customer orders is a random variable with the following cdf:

$$F_X(x) = \begin{cases} 0, & x < \frac{1}{8} \\ 0.2, & \frac{1}{8} \le x < \frac{1}{4} \\ 0.9, & \frac{1}{4} \le x < \frac{3}{8} \\ 1 & x \ge \frac{3}{8} \end{cases}$$

Determine the following probabilities:

(a) $P[X \le 1/18]$

(b) $P[X \le 1/4]$

(c) $P[X \le 5/16]$

(d) $P[X > 1/4]$

(e) $P[X \le 1/2]$

[Montgomery and Runger, 2010, Q3-42]

Solution:

(a) $P[X \le 1/18] = F_X(1/18) = \boxed{0}$ because $\frac{1}{18} < \frac{1}{8}$.

(b) $P[X \le 1/4] = F_X(1/4) = \boxed{0.9}$.

(c) $P[X \le 5/16] = F_X(5/16) = \boxed{0.9}$ because $\frac{1}{4} < \frac{5}{16} < \frac{3}{8}$.

(d) $P[X > 1/4] = 1 - P[X \le 1/4] = 1 - F_X(1/4) = 1 - 0.9 = \boxed{0.1}$.

(e) $P[X \le 1/2] = F_X(1/2) = \boxed{1}$ because $\frac{1}{2} > \frac{3}{8}$.

Alternatively, we can also derive the pmf first and then calculate the probabilities.

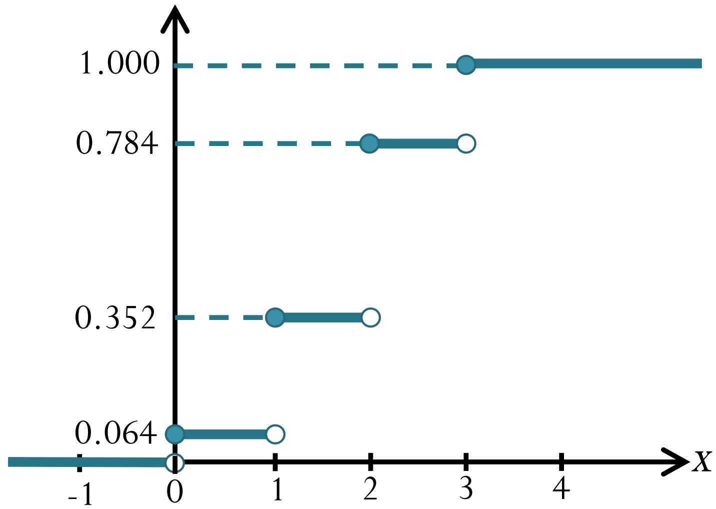


Figure 6.1: CDF of X for Problem 2

Problem 2. [M2011/1] The cdf of a random variable X is plotted in Figure 6.1.

(a) Find the pmf $p_X(x)$.

(b) Find the family to which X belongs. (Uniform, Bernoulli, Binomial, Geometric, Poisson, etc.)

Solution:

(a) For discrete random variable, $P[X = x]$ is the jump size at x on the cdf plot. In this problem, there are four jumps at 0, 1, 2, 3.

- $P[X=0]$ = the jump size at $0 = 0.064 = \frac{64}{1000} = (4/10)^3 = (2/5)^3$.
- $P[X=1]$ = the jump size at $1 = 0.352 - 0.064 = 0.288$.
- $P[X=2]$ = the jump size at $2 = 0.784 - 0.352 = 0.432$.
- $P[X=3]$ = the jump size at $3 = 1 - 0.784 = 0.216 = (6/10)^3$.

In conclusion,

$$\boxed{p_X(x) = \begin{cases} 0.064, & x = 0, \\ 0.288, & x = 1, \\ 0.432, & x = 2, \\ 0.216, & x = 3, \\ 0, & \text{otherwise.} \end{cases}}$$

(b) Among all the pmf that we discussed in class, only one can have support $= \{0, 1, 2, 3\}$ with unequal probabilities. This is the $\boxed{\text{binomial}}$ pmf. To check that it really is Binomial, recall that the pmf for binomial X is given by $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$ for $x = 0, 1, 2, \ldots, n$. Here, $n = 3$. Furthermore, observe that $p_X(0) = (1-p)^n$. By comparing $p_X(0)$ with what we had in part (a), we have $1 - p = 2/5$ or $p = 3/5$. For $x = 1, 2, 3$, plugging in $p = 3/5$ and $n = 3$ in to $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$ gives the same values as what we had in part (a). So, X is a binomial RV.

Problem 3. Arrivals of customers at the local supermarket are modeled by a Poisson process with a rate of $\lambda = 2$ customers per minute. Let M be the number of customers arriving between 9:00 and 9:05. What is the probability that $M < 2$?

Solution: Here, we are given that $M \sim \mathcal{P}(\alpha)$ where $\alpha = \lambda T = 2 \times 5 = 10$. Recall that, for $M \sim \mathcal{P}(\alpha)$, we have

$$P[M = m] = \begin{cases} e^{-\alpha} \frac{\alpha^m}{m!}, & m \in \{0, 1, 2, 3, \ldots\} \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} P[M < 2] &= P[M = 0] + P[M = 1] = e^{-\alpha} \frac{\alpha^0}{0!} + e^{-\alpha} \frac{\alpha^1}{1!} \\ &= e^{-\alpha}(1 + \alpha) = e^{-10}(1 + 10) = 11e^{-10} \approx 5 \times 10^{-4}. \end{aligned}$$

Problem 4. When n is large, binomial distribution Binomial(n, p) becomes difficult to compute directly because of the need to calculate factorial terms. In this question, we will consider an approximation when the value of p is close to 0. In such case, the binomial can be approximated[1] by the Poisson distribution with parameter $\alpha = np$. For this approximation to work, we will see in this exercise that n does not have to be very large and p does not need to be very small.

(a) Let $X \sim$ Binomial$(12, 1/36)$. (For example, roll two dice 12 times and let X be the number of times a double 6 appears.) Evaluate $p_X(x)$ for $x = 0, 1, 2$.

(b) Compare your answers part (a) with its Poisson approximation.

[1] More specifically, suppose X_n has a binomial distribution with parameters n and p_n. If $p_n \to 0$ and $np_n \to \alpha$ as $n \to \infty$, then

$$P[X_n = k] \to e^{-\alpha} \frac{\alpha^k}{k!}.$$

(c) Compare `MATLAB` plots of $p_X(x)$ in part (a) and the pmf of $\mathcal{P}(np)$.

Solution:

(a) For Binomial(n,p) random variable,

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0,1,2,\dots,n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we are given that $n = 12$ and $p = \frac{1}{36}$. Plugging in $x = 0,1,2$, we get $\boxed{0.7132, 0.2445, 0.0384}$, respectively

(b) A Poisson random variable with parameter $\alpha = np$ can approximate a Binomial(n,p) random variable when n is large and p is small. Here, with $n = 12$ and $p = \frac{1}{36}$, we have $\alpha = 12 \times \frac{1}{36} = \frac{1}{3}$. The Poisson pmf at $x = 0,1,2$ is given by $e^{-\alpha}\frac{\alpha^x}{x!} = e^{-1/3}\frac{(1/3)^x}{x!}$. Plugging in $x = 0,1,2$ gives $\boxed{0.7165, 0.2388, 0.0398}$, respectively.

(c) See Figure 6.2. Note how close they are!

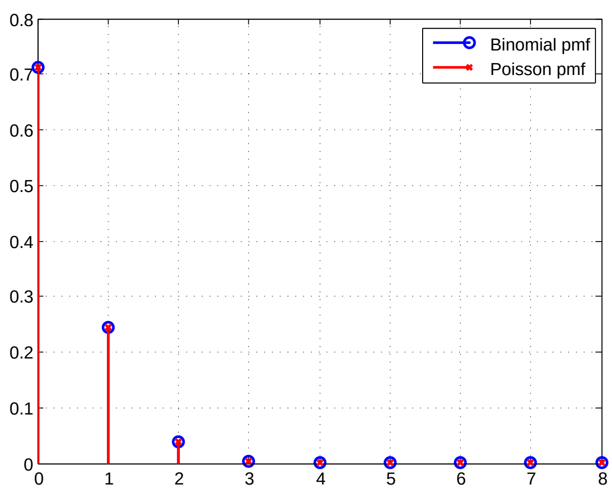


Figure 6.2: Poisson Approximation

Problem 5. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson pmf. (For simplicity, exclude birthdays on February 29.) [Bertsekas and Tsitsiklis, 2008, Q2.2.2]

Solution: Let N be the number of guests that has the same birthday as you. We may think of the comparison of your birthday with each of the guests as a Bernoulli trial. Here, there are 500 guests and therefore we are considering $n = 500$ trials. For each trial, the (success) probability that you have the same birthday as the corresponding guest is $p = \frac{1}{365}$. Then, this $N \sim \text{Binomial}(n, p)$.

(a) Binomial: $P[N = 1] = np^1(1-p)^{n-1} \approx 0.348$.

(b) Poisson: $P[N = 1] = e^{-np}\frac{(np)^1}{1!} \approx 0.348$.

Extra Questions

Here are some questions for those who want extra practice.

Problem 6. A sample of a radioactive material emits particles at a rate of 0.7 per second. Assuming that these are emitted in accordance with a Poisson distribution, find the probability that in one second

(a) exactly one is emitted,

(b) more than three are emitted,

(c) between one and four (inclusive) are emitted

[Applebaum, 2008, Q5.27].

Solution: Let X be the number or particles emitted during the one second under consideration. Then $X \sim \mathcal{P}(\alpha)$ where $\alpha = \lambda T = 0.7 \times 1 = 0.7$.

(a) $P[X = 1] = e^{-\alpha}\frac{\alpha^1}{1!} = \alpha e^{-\alpha} = 0.7e^{-0.7} \approx 0.3477$.

(b) $P[X > 3] = 1 - P[X \le 3] = 1 - \sum_{k=0}^{3} e^{-0.7}\frac{0.7^k}{k!} \approx 0.0058$.

(c) $P[1 \le X \le 4] = \sum_{k=1}^{4} e^{-0.7}\frac{0.7^k}{k!} \approx 0.5026$.

Problem 7 (M2011/1)**.** You are given an unfair coin with probability of obtaining a head equal to $1/3,000,000,000$. You toss this coin 6,000,000,000 times. Let A be the event that you get "tails for all the tosses". Let B be the event that you get "heads for all the tosses".

(a) Approximate $P(A)$.

(b) Approximate $P(A \cup B)$.

Solution: Let N be the number of heads among the n tosses. Then, $N \sim \mathcal{B}(n,p)$. Here, we have small $p = 1/3 \times 10^9$ and large $n = 6 \times 10^9$. So, we can apply Poisson approximation. In other words, $\mathcal{B}(n,p)$ is well-approximated by $\mathcal{P}(\alpha)$ where $\alpha = np = 2$.

(a) $P(A) = P[N=0] = e^{-2}\frac{2^0}{0!} = \frac{1}{e^2} \approx 0.1353$.

(b) $P(A \cup B) = P[N=0] + P[N=n] = e^{-2}\frac{2^0}{0!} + e^{-2}\frac{2^{6\times10^9}}{(6\times10^9)!}$. The second term is extremely small compared to the first one. Hence, $P(A \cup B)$ is approximately the same as $P(A)$.

Problem 8. In one of the New York state lottery games, a number is chosen at random between 0 and 999. Suppose you play this game 250 times. Use the Poisson approximation to estimate the probability that you will never win and compare this with the exact answer.

Solution:[Durrett, 2009, Q2.41] Let W be the number of wins. Then, $W \sim \text{Binomial}(250, p)$ where $p = 1/1000$. Hence,

$$P[W=0] = \binom{250}{0} p^0 (1-p)^{250} \approx 0.7787.$$

If we approximate W by $\Lambda \sim \mathcal{P}(\alpha)$. Then we need to set

$$\alpha = np = \frac{250}{1000} = \frac{1}{4}.$$

In which case,

$$P[\Lambda = 0] = e^{-\alpha}\frac{\alpha^0}{0!} = e^{-\alpha} \approx 0.7788$$

which is very close to the answer from direct calculation.