ECS315 2014 Quiz 4 Solution

Continue from Quiz 3 and Ex. 8.13

Suppose a RV X has pmf $p_X(x) = \begin{cases} \frac{6}{11x}, & x = 1,2,3, \\ 0, & \text{otherwise}. \end{cases}$

① Find $\mathbb{E}X = \sum_x x\, p_X(x) = \sum_x \cancel{x} \frac{6}{11\cancel{x}} = 3 \times \frac{6}{11} = \frac{18}{11}$

There are three values in the support of X. Therefore, the sum here has three x-values.

② Suppose $Y = (X-2)^2$. a) Find $p_Y(y)$

x	$y=(x-2)^2$
1	1
2	0
3	1

$P[Y=0] = P[X=2] = \frac{6}{11\times 2} = \frac{3}{11}$

$P[Y=1] = P[X=1] + P[X=3] = 1 - \frac{3}{11} = \frac{8}{11}$

$$p_Y(y) = \begin{cases} 3/11, & y=0, \\ 8/11, & y=1, \\ 0, & \text{otherwise} \end{cases}$$

b) Find $\mathbb{E}Y$ from $p_Y(y)$

$= 0 \times \frac{3}{11} + 1 \times \frac{8}{11} = \frac{8}{11}$

c) Find $\mathbb{E}Y$ via LOTUS

$= \sum_x g(x) p_X(x) = \sum_x (x-2)^2 p_X(x)$

$= 1 \times \frac{6}{11} + 0 \times \frac{3}{11} + 1 \times \frac{2}{11} = \frac{8}{11}$

③ Suppose $Z = Y + 3(X-2) + 1$. Find $\mathbb{E}Z$.

$$\mathbb{E}Z = \mathbb{E}\left[Y + 3(X-2)+1\right] = \underbrace{\mathbb{E}Y}_{\frac{8}{11}} + \underbrace{\mathbb{E}[3X-6]}_{\substack{3\mathbb{E}X - 6 \\ 3\times\frac{18}{11} - 6}} + \underbrace{\mathbb{E}[1]}_{1} = \frac{7}{11}$$

Alternatively,

$p_X(x)$	x	$y=(x-2)^2$	$z = y + 3(x-2)+1$
6/11	1	1	$1 + 3(-1) + 1 = -1$
3/11	2	0	$0 + 3(0) + 1 = 1$
2/11	3	1	$1 + 3(1) + 1 = 5$

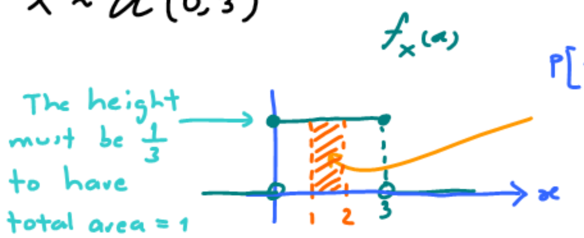
$$p_Z(z) = \begin{cases} 6/11, & z=-1, \\ 3/11, & z=1. \\ 2/11, & z=5, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, $\mathbb{E}Z = \frac{6}{11}\times(-1) + \frac{3}{11}\times(1) + \frac{2}{11}\times 5 = \frac{-6+3+10}{11} = \frac{7}{11}$

ECS315 2014 Quiz 5 Solution

Evaluate $P[1 < X < 2]$ for each of the following RVs

(a) $X \sim \mathcal{U}(0,3)$



$P[1 \le X \le 2] = 1 \times \frac{1}{3} = \frac{1}{3} \approx 0.3333$

Alternatively, the cdf of $\mathcal{U}(0,3)$ is

$$F_X(x) = \begin{cases} \frac{x-0}{3-0}, & 0 \le x \le 3, \\ 0, & \text{otherwise} \end{cases}$$

Therefore, $P[1 \le X \le 2] = F_X(2) - F_X(1) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

(b) $X \sim \mathcal{E}(3)$ (λ)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 3e^{-3x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$P[1 \le X \le 2] = \int_1^2 f_X(x)\,dx = \int_1^2 3e^{-3x}\,dx = 3\frac{e^{-3x}}{-3}\Big|_1^2 = e^{-3} - e^{-6} \approx 0.0473$$

Alternatively, the cdf of $\mathcal{E}(3)$ is

$$F_X(x) = \begin{cases} 1 - e^{-3x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $P[1 \le X \le 2] = F_X(2) - F_X(1) = (1 - e^{-6}) - (1 - e^{-3}) = e^{-3} - e^{-6}$

(c) $X \sim \mathcal{N}(0,1)$

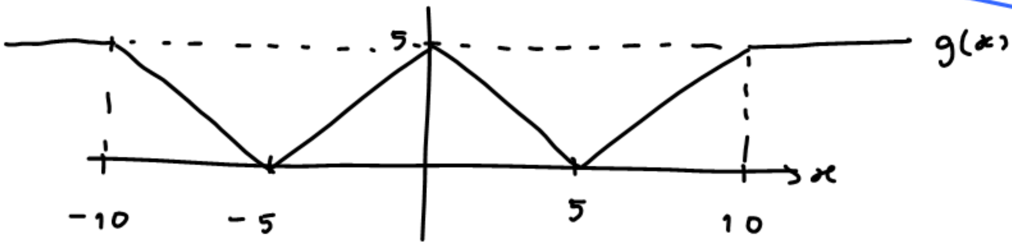
$$P[1 \le X \le 2] = \Phi(2) - \Phi(1) \approx 0.97725 - 0.8413 \approx 0.1359$$

(d) $X \sim \mathcal{N}(1,3)$

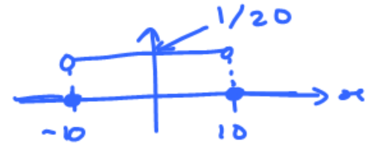
$$P[1 \le X \le 2] = \Phi\left(\frac{2-1}{\sqrt{3}}\right) - \Phi\left(\frac{1-1}{\sqrt{3}}\right) = \Phi\left(\frac{1}{\sqrt{3}}\right) - \Phi(0) \approx \Phi(0.58) - 0.5$$

$$\approx 0.7190 - 0.5 = 0.2190$$

Quiz #6. $X \sim \mathcal{U}(-10,10)$. Let $Y = g(X)$.

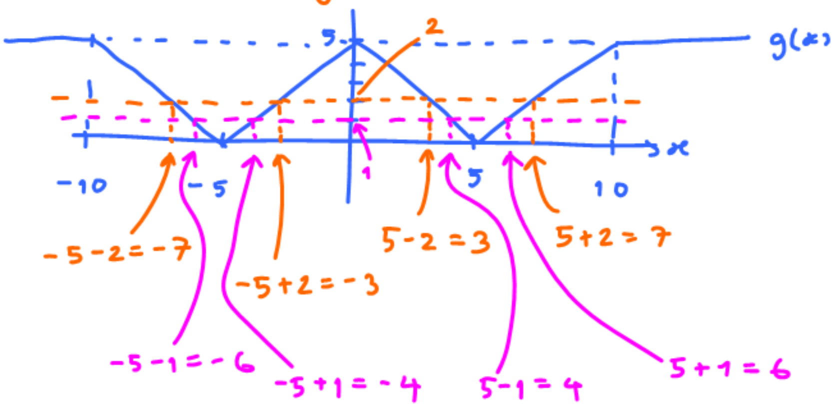


$$f_X(x) = \begin{cases} \frac{1}{20}, & -10 < x < 10, \\ 0, & \text{otherwise} \end{cases}$$



(a) Find $f_Y(2)$.

① Find root(s) of $g(x) = 2$. $\Rightarrow x = \pm 3, \pm 7$



② Find slopes: at $x = -3, 7$, $g'(x) = 1$.

at $x = 3, -7$, $g'(x) = -1$.

③ $$f_Y(2) = \frac{f_X(-7)}{|g'(-7)|} + \frac{f_X(-3)}{|g'(-3)|} + \frac{f_X(3)}{|g'(3)|} + \frac{f_X(7)}{|g'(7)|} = \frac{1/20}{|-1|} + \frac{1/20}{|1|} + \frac{1/20}{|-1|} + \frac{1/20}{|1|}$$

$$= \frac{1}{20} \times 4 = \frac{1}{5}.$$

(b) Find $f_Y(1)$.

Find root(s) of $g(x) = 1 \Rightarrow x = \pm 4, \pm 6$

$$f_Y(1) = \frac{f_X(-6)}{|g'(-6)|} + \frac{f_X(-4)}{|g'(-4)|} + \frac{f_X(4)}{|g'(4)|} + \frac{f_X(6)}{|g'(6)|}$$

$$= \frac{1/20}{|-1|} + \frac{1/20}{|1|} + \frac{1/20}{|-1|} + \frac{1/20}{|1|} = \frac{1}{20} \times 4 = \frac{1}{5}$$

(c) Repeat (a) and (b) but use $X \sim \mathcal{E}(1)$. $f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise} \end{cases}$

(a) $$f_Y(2) = \frac{\cancel{f_X(-7)}^{\,0}}{|g'(-7)|} + \frac{\cancel{f_X(-3)}^{\,0}}{|g'(-3)|} + \frac{f_X(3)}{|g'(3)|} + \frac{f_X(7)}{|g'(7)|} = \frac{e^{-3}}{1} + \frac{e^{-7}}{1} \approx 0.0507$$

(b) $$f_Y(1) = \frac{\cancel{f_X(-6)}^{\,0}}{|g'(-6)|} + \frac{\cancel{f_X(-4)}^{\,0}}{|g'(-4)|} + \frac{f_X(4)}{|g'(4)|} + \frac{f_X(6)}{|g'(6)|} = \frac{e^{-4}}{1} + \frac{e^{-6}}{1} \approx 0.0208$$

Quiz 7:

Find the marginal pmfs $p_X(x)$ and $p_Y(y)$ of two RVs X and Y whose joint pmf matrix is

$$
P_{X,Y} = \begin{array}{c|cccc} x \backslash y & 0 & 1 & 2 & 3 \\ \hline 0 & 0.1 & 0 & 0.2 & 0 \\ 1 & 0 & 0.5 & 0 & 0 \\ 2 & 0 & 0.1 & 0.1 & 0 \end{array}
$$

sum along each row

$$
\begin{array}{c|cccc|c} x \backslash y & 0 & 1 & 2 & 3 & \\ \hline 0 & 0.1 & 0 & 0.2 & 0 & \xrightarrow{\Sigma} 0.3 \\ 1 & 0 & 0.5 & 0 & 0 & \xrightarrow{\Sigma} 0.5 \\ 2 & 0 & 0.1 & 0.1 & 0 & \xrightarrow{\Sigma} 0.2 \\ \hline & \Sigma\downarrow & \Sigma\downarrow & \Sigma\downarrow & \Sigma\downarrow & \\ & 0.1 & 0.6 & 0.3 & 0 & \end{array}
$$

sum along each column

$$
\Rightarrow \boxed{p_X(x) = \begin{cases} 0.3, & x = 0, \\ 0.5, & x = 1, \\ 0.2, & x = 2, \\ 0, & \text{otherwise.} \end{cases}}
$$

$$
\Downarrow
$$

$$
\boxed{p_Y(y) = \begin{cases} 0.1, & y = 0, \\ 0.6, & y = 1, \\ 0.3, & y = 2, \\ 0, & \text{otherwise.} \end{cases}}
$$