Tutorial on Sep 27, 2013 - part 1

Friday, September 27, 2013 9:05 AM

$\mathbb{E}\left[(\underbrace{Y - 3X + 5}_{Z})^2 \right]$

$\sigma_X, \sigma_Y, \rho_{X,Y}$

$\operatorname{Var} Y = \sigma_Y^2$

$\operatorname{cov}[X,Y] = \sigma_X \times \sigma_Y \times \rho_{X,Y}$

$\operatorname{Var} X = \sigma_X^2$

$\mathbb{E}[Y - 3X + 5] = 1 \Rightarrow \mathbb{E} Z = 1 = c$

$= \mathbb{E}[Z^2] = \operatorname{Var}[Z] + c^2$

$= \operatorname{Var}[Y - 3X + 5] = \operatorname{Var}[Y - 3X] = \operatorname{Var}[Y + (-3X)]$

$\operatorname{Var}[W + \alpha] = \operatorname{Var}[W]$

$\operatorname{Var}[aX]$
$= a^2 \operatorname{Var} X$

$= (-3)^2 \operatorname{Var} X$

$= \operatorname{Var}[Y] + \operatorname{Var}[-3X] + 2\underbrace{\operatorname{cov}[Y, -3X]}_{= -3 \operatorname{cov}[Y,X]}$

$\operatorname{Var}[X+Y] = \operatorname{Var} X + \operatorname{Var} Y + 2 \operatorname{cov}[X,Y]$

$\operatorname{cov}[aX, bY] = \mathbb{E}\left[(aX - a\mathbb{E}X)(bY - b\mathbb{E}Y) \right]$

$= ab\, \underbrace{\mathbb{E}\left[(X - \mathbb{E}X)(Y - \mathbb{E}Y) \right]}_{\operatorname{cov}[X,Y]}$

$\operatorname{Var}[aX] = \operatorname{cov}[aX, aX] = a \times a \times \operatorname{cov}[X, X] = a^2 \operatorname{Var} X$

$X \sim \text{binomial}(n, p)$

$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$\mathbb{E} X = \sum^{n} k \binom{n}{k} p^k (1-p)^{n-k} \quad = np$

$$\mathbb{E}X = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$$

$$\mathbb{E}[X^2] = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} = np\left(p(n-1)+1\right)$$

$$\mathbb{E}[X^3] = \sum_{k=0}^{n} k^3 \binom{n}{k} p^k (1-p)^{n-k} = np + 3n(n-1)p^2 + n(n-1)(n-2)p^3$$

Derivations of the formulas above:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\Big\downarrow \frac{\partial}{\partial x}$$

$$n(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1} y^{n-k}$$

$$\Big\downarrow \times x$$

$$nx(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k} y^{n-k}$$

$$\Big\downarrow \frac{\partial}{\partial x}$$

$$n(x+y)^{n-1} + nx(n-1)(x+y)^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k^2 x^{k-1} y^{n-k}$$

$$\Big\downarrow \times x$$

$$nx(x+y)^{n-1} + n(n-1)x^2(x+y)^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k^2 x^{k} y^{n-k}$$

$$\Big\downarrow \frac{\partial}{\partial x}$$

$$\begin{aligned} &n(x+y)^{n-1} + nx(n-1)(x+y)^{n-2} \\ &+ n(n-1)2x(x+y)^{n-2} + n(n-1)x^2(n-2)(x+y)^{n-3} \end{aligned} = \sum_{k=0}^{n} \binom{n}{k} k^3 x^{k-1} y^{n-k}$$

$$\Big\downarrow \times x$$

$$nx(x+y)^{n-1} + 3n(n-1)x^2(x+y)^{n-2} + n(n-1)(n-2)x^3(x+y)^{n-3} = \sum_{k=0}^{n} \binom{n}{k} k^3 x^{k} y^{n-k}$$

$$n x (x+y)^{n-1} + 3n(n-1)x^2(x+y)^{n-2} + n(n-1)(n-2)x^3(x+y)^{n-3} = \sum_{k=0}^{n} \binom{n}{k} k^3 x^k y^{n-k}$$

Another way to think about the expected value and variance of binomial RV

$$Y \sim \text{binomial}(n,p)$$

Observe that

$$Y = X_1 + X_2 + \cdots + X_n \qquad X_i \overset{iid}{\sim} \text{bernoulli}(p)$$

$$\mathbb{E}X_i = 0 \times (1-p) + 1p = p$$

$$\mathbb{E}Y = \mathbb{E}X_1 + \mathbb{E}X_2 + \cdots + \mathbb{E}X_n$$

$$= p + p + \cdots + p = np$$

$$\text{Var}\, X_i = p - p^2 = p(1-p)$$

$$\mathbb{E}[X_i^2] = p$$

$$\text{Var}\, Y = \text{Var}[X_1 + X_2 + \cdots + X_n]$$

Let's try to work with $n=2$ first:

$$Y \sim \text{binomial}(2,p)$$

$$Y = X_1 + X_2$$

$$\text{Var}\, Y = \text{Var}(X_1 + X_2) = \text{Var}\, X_1 + \text{Var}\, X_2 + 2\,\underbrace{\cancel{\text{Cov}}(X_1, X_2)}_{0}$$

$$= 2p(1-p)$$

Now, when $n=3$,

these two RVs are independent

$$\text{Var}[\underbrace{X_1 + X_2} + \underbrace{X_3}] = \text{Var}(X_1 + X_2) + \text{Var}\, X_3 = 3p(1-p)$$

So, in general,

$$\text{Var}\left[\sum_{k=1}^{n} X_k\right] = np(1-p) = np - np^2$$

Conclusion: If we have i.i.d. $X_1, X_2, \ldots, X_n$,

(independent)

(uncorrelated)

$$\mathrm{Var}\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} \mathrm{Var}[X_k].$$