ECS 315: Probability and Random Processes	2013/1

HW 9 — Due: Sep 13

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Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

θ Θ

x X

$g(\theta) = 5\cos(7t+\theta)$

$\mathbb{E}[X] = \mathbb{E}[g(\Theta)]$

$= \int_0^{2\pi} 5\cos(7t+\theta)\frac{1}{2\pi}\,d\theta$

$= 0$

(b) Consider another random variable Y defined by

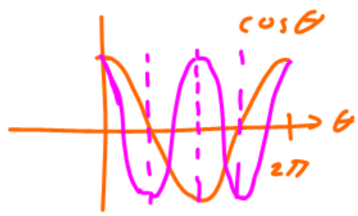
$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

$= \frac{25}{2}\Big(\cos(14(t_1+t_2)+2\Theta) + \cos(7(t_1-t_2))\Big)$

$\mathbb{E}Y = \frac{25}{2}\Big(\cos(7(t_1-t_2)) + \frac{1}{2\pi}\int_0^{2\pi}\cos(14(t_1+t_2)+2\theta)\,d\theta\Big)$

0



Problem 2 (Yates and Goodman, 2005, Q3.4.5)**.** X is a continuous uniform $(-5, 5)$ random varible.

(a) What is its pdf $f_X(x)$?

(b) What is its cdf $F_X(x)$?

(c) What is $\mathbb{E}[X]$?

(d) What is $\mathbb{E}[X^5]$?

$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

(e) What is $\mathbb{E}\left[e^X\right]$?

Problem 3. A random variable X is a Gaussian random variable if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2},$$

for some constants m and σ. Furthermore, when a Gaussian random variable has $\cancel{m =}$ $m = 0$ $\sigma = 1$, we say that it is a standard Gaussian random variable. There is no closed-form expression for the cdf of the standard Gaussian random variable. The cdf itself is denoted by Φ and its values (or its complementary values $Q(\cdot) = 1 - \Phi(\cdot)$) are traditionally provided by a table. We refer to this kind of table as the Φ table. Examples of such tables are Table 3.1 and Table 3.2 in [Y&G].

Suppose Z is a standard Gaussian random variable.

(a) Use the Φ table to find the following probabilities:

(i) $P[Z < 1.52]$ $= P[Z \le 1.52] = F_Z(1.52) = \Phi(1.52) = \ldots$

(ii) $P[Z < -1.52]$ ① $= P[Z > 1.52] = 1 - P[Z \le 1.52] = 1 - F_Z(1.52) = 1 - \Phi(1.52) = \ldots$

(iii) $P[Z > 1.52]$ ② $= P[Z \le -1.52] = F_Z(-1.52) = \Phi(-1.52) = 1 - \Phi(1.52)$

(iv) $P[Z > -1.52]$

(v) $P[-1.36 < Z < 1.52]$

(b) Use the Φ table to find the value of c that satisfies each of the following relation.

(i) $P[Z > c] = 0.14$ $= 1 - P[Z \le c] = 1 - \Phi(c) = 0.14$

$\Phi(c) = 0.86$

(ii) $P[-c < Z < c] = 0.95$

Problem 4. The peak temperature T, as measured in degrees Fahrenheit, on a July day in New Jersey is a $\mathcal{N}(85, 100)$ random variable.

Remark: Do not forget that, for our class, the second parameter in $\mathcal{N}(\cdot,\cdot)$ is the variance (not the standard deviation).

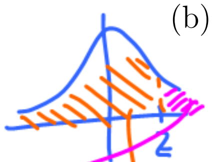
(a) Express the cdf of T in terms of the Φ function

Hint: Recall that the cdf of a random variable T is given by $F_T(t) = P[T \le t]$. For $T \sim \mathcal{N}(m, \sigma^2)$,

$$F_T(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx. \quad = \Phi\left(\frac{t-m}{\sigma}\right)$$

The Φ function, which is the cdf of the standard Gaussian random variable, is given by

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$



(b) Express each of the following probabilities in terms of the Φ function(s). Make sure that the arguments of the Φ functions are positive. (Positivity is required so that we can directly use the Φ/Q tables to evaluate the probabilities.)

(i) $P[T > 100]$

$F_T(100) - F_T(70) = \Phi\left(\frac{100-85}{10}\right) - \Phi\left(\frac{70-85}{10}\right)$

$Q(z) + \Phi(z) = 1$

(ii) $P[T < 60]$

(iii) $P[70 \le T \le 100]$

$= \Phi(1.5) - \Phi(-1.5) = \Phi(1.5) - (1 - \Phi(1.5))$

$= 2\Phi(1.5) - 1$

$= 2(1 - Q(1.5)) - 1 = 1 - 2Q(1.5)$

(c) Express each of the probabilities in part (b) in terms of the Q function(s). Again, make sure that the arguments of the Q functions are positive.

(d) Evaluate each of the probabilities in part (b) using the Φ/Q tables.

(e) Observe that Table 3.1 stops at $z = 2.99$ and Table 3.2 starts at $z = 3.00$. Why is it better to give a table for $Q(z)$ instead of $\Phi(z)$ when z is large?

e^{-5} 0 1

Problem 5. (Function of Continuous Random Variable) Let $X \sim \mathcal{E}(5)$ and $Y = 2/X$. Find

(a) $F_Y(y)$.

(b) $f_Y(y)$.

(c) $\mathbb{E}Y$ $= \int_{-\infty}^{\infty} \frac{2}{x} f_X(x)\, dx = \int_0^{\infty} \left(\frac{2}{x} 5 e^{-5x}\right) dx = \int_0^1 dx + \int_1^{\infty} dx > \int_0^1 dx > \int_0^1 \frac{10}{x} e^{-5} dx = \infty$

Hint: Because $\frac{d}{dy} e^{-\frac{10}{y}} = \frac{10}{y^2} e^{-\frac{10}{y}} > 0$ for $y \neq 0$. We know that $e^{-\frac{10}{y}}$ is an increasing function on our range of integration. In particular, consider $y > 10/\ln(2)$. Then, $e^{-\frac{10}{y}} > \frac{1}{2}$. Hence,

$$\int_0^{\infty} \frac{10}{y} e^{-\frac{10}{y}} dy > \int_{10/\ln 2}^{\infty} \frac{10}{y} e^{-\frac{10}{y}} dy > \int_{10/\ln 2}^{\infty} \frac{10}{y}\frac{1}{2} dy = \int_{10/\ln 2}^{\infty} \frac{5}{y} dy. = \infty$$

$X \sim \mathcal{E}(5) \qquad Y = \frac{2}{X} \qquad y < 0 \quad F_Y(y) = 0$

$y = 0 \quad F_Y(y) = 0$

For $y > 0$,

$$F_Y(y) = P[Y \leqslant y] = P\left[\frac{2}{X} \leqslant y\right] = P\left[X \geqslant \frac{2}{y}\right] = \exp\left(-5 \times \frac{2}{y}\right) = e^{-\frac{10}{y}}$$

$$P[X > x] = P[X \geqslant x] = \begin{cases} e^{-\lambda x}, & x > 0, \\ 1, & \text{otherwise.} \end{cases}$$

$P[X \geqslant -8]$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$F_Y(y) = \begin{cases} e^{-\frac{10}{y}}, & y > 0 \\ 0, & y \leqslant 0 \end{cases} \xrightarrow{\frac{d}{dy}} f_Y(y) = \begin{cases} \frac{10}{y^2} e^{-\frac{10}{y}}, & y > 0 \\ 0 & y \leqslant 0 \end{cases}$$

Method 2:

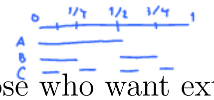
$$f_Y(y)|\Delta y| = f_X(x)|\Delta x| \qquad 5e^{-5x}$$

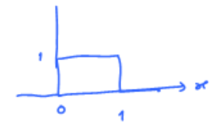
$$f_Y(y) = f_X(x)\left|\frac{\Delta x}{\Delta y}\right| = f_X(x)\left|\frac{dx}{dy}\right| =$$

$$y = \frac{2}{x} \qquad x = \frac{2}{y}$$

$$\frac{dx}{dy} = -\frac{2}{y^2}$$

Extra Questions





Here are some questions for those who want extra practice.

$f_X(x) =$

Problem 6. Let X be a uniform random variable on the interval $[0, 1]$. Set

$$A = \left[0, \frac{1}{2}\right), \quad B = \left[0, \frac{1}{4}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right), \quad \text{and } C = \left[0, \frac{1}{8}\right) \cup \left[\frac{1}{4}, \frac{3}{8}\right) \cup \left[\frac{1}{2}, \frac{5}{8}\right) \cup \left[\frac{3}{4}, \frac{7}{8}\right).$$

① $P(D)P(E)$ ② $P(E)P(F)$ ③ $P(D)P(F)$

$P(D) = 1/2 = P(E) = P(F)$ $P(D \cap E) = \frac{1}{4} = P(E \cap F) = P(D \cap F)$ $P(D \cap E \cap F) = \frac{1}{8}$

Are the events $\underbrace{[X \in A]}_{D}$, $\underbrace{[X \in B]}_{E}$, and $\underbrace{[X \in C]}_{F}$ independent? ④ $P(D)P(E)P(F)$

Yes.

Problem 7 (Q3.5.6)**.** Solve this question using Table 3.1 and/or Table 3.2 from [Yates & Goodman, 2005]:

A professor pays 25 cents for each blackboard error made in lecture to the student who points out the error. In a career of n years filled with blackboard errors, the total amount in dollars paid can be approximated by a Gaussian random variable Y_n with expected value $40n$ and variance $100n$.

(a) What is the probability that Y_{20} exceeds 1000?

(b) How many years n must the professor teach in order that $P[Y_n > 1000] > 0.99$?