ECS 315: Probability and Random Processes **2013/1**

HW Solution 4 — Due: July 19

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Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Show that if A and B are independent events, then so are A^c and B^c.

Solution: To show that two events C_1 and C_2 are independent, we need to show that $P(C_1 \cap C_2) = P(C_1)P(C_2)$.

From set theory, we know that $A^c \cap B^c = (A \cup B)^c$. Therefore,

$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)$$

where, for the last equality, we use

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

which is discussed in class.

Because $A \perp\!\!\!\perp B$, we have

$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c). \end{aligned}$$

An extra problem at the end of this assignment considers the other cases.

Problem 2. A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases. A specific design is randomly generated by the Web server when you visit the site. Let A denote the event that the design color is red and let B denote the event that the font size is not the smallest one.

(a) Use classical probability to evaluate $P(A)$, $P(B)$ and $P(A \cap B)$. Show that the two events A and B are independent by checking whether $P(A \cap B) = P(A)P(B)$.

(b) Using the values of $P(A)$ and $P(B)$ from the previous part and the fact that $A \perp\!\!\!\perp B$, calculate the following probabilities.

(i) $P(A \cup B)$

(ii) $P(A \cup B^c)$

(iii) $P(A^c \cup B^c)$

[Montgomery and Runger, 2010, Q2-84]

Solution:

(a) There are

$$|\Omega| = 4 \times 3 \times 5 \times 3 \times 5 \tag{4.1}$$

possible designs. The number of designs whose color is red is given by

$$|A| = 1 \times 3 \times 5 \times 3 \times 5.$$

Note that the "4" in (4.1) is replaced by "1" because we only consider one color (red). Therefore,

$$P(A) = \frac{1 \times 3 \times 5 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{1}{4}.}$$

Similarly, $|B| = 4 \times 3 \times 4 \times 3 \times 5$ where the "5" in the middle of (4.1) is replaced by "4" because we can't use the smallest font size. Therefore,

$$P(B) = \frac{4 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{4}{5}.}$$

For the event $A \cap B$, we replace "4" in (4.1) by "1" because we need red color and we replace "5" in the middle of (4.1) by "4" because we can't use the smallest font size. This gives

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{1 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \frac{1 \times 4}{4 \times 5} = \boxed{\frac{1}{5}} = 0.2.$$

Because $P(A \cap B) = P(A)P(B)$, the events A and B are independent.

(b)

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap C) = \frac{1}{4} + \frac{4}{5} - \frac{1}{5} = \boxed{\frac{17}{20} = 0.85.}$

(ii) $P(A^c \cup B^c) = 1 - P\left((A^c \cup B^c)^c\right) = 1 - P(A \cap B) = 1 - 0.2 = \boxed{0.8.}$

(iii) $P(A \cup B^c) = 1 - P\left((A \cup B^c)^c\right) = 1 - P(A^c \cap B)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$. Hence, $P(A^c \cup B^c) = 1 - P(A^c)P(B) = 1 - \frac{3}{4}\frac{4}{5} = \frac{2}{5} = \boxed{0.4.}$

Problem 3. In this question, each experiment has equiprobable outcomes.

(a) Let $\Omega = \{1,2,3,4\}$, $A_1 = \{1,2\}$, $A_2 = \{1,3\}$, $A_3 = \{2,3\}$.

(i) Determine whether $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.

(ii) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

(iii) Are A_1, A_2, and A_3 independent?

(b) Let $\Omega = \{1,2,3,4,5,6\}$, $A_1 = \{1,2,3,4\}$, $A_2 = A_3 = \{4,5,6\}$.

(i) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

(ii) Check whether $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.

(iii) Are A_1, A_2, and A_3 independent?

Solution:

(a) We have $P(A_i) = \frac{1}{2}$ and $P(A_i \cap A_j) = \frac{1}{4}$.

(i) $P(A_i \cap A_j) = P(A_i)P(A_j)$ for any $i \neq j$.

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$. Hence, $P(A_1 \cap A_2 \cap A_3) = 0$, which is *not* the same as $P(A_1)P(A_2)P(A_3)$.

(iii) No.

Remark: This counter-example shows that pairwise independence does not imply independence.

(b) We have $P(A_1) = \frac{4}{6} = \frac{2}{3}$ and $P(A_2) = P(A_3) = \frac{3}{6} = \frac{1}{2}$.

(i) $A_1 \cap A_2 \cap A_3 = \{4\}$. Hence, $P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}$.
$P(A_1)P(A_2)P(A_3) = \frac{2}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{6}$.
Hence, $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

(ii) $P(A_2 \cap A_3) = P(A_2) = \frac{1}{2} \neq P(A_2)P(A_3)$
$P(A_1 \cap A_2) = p(4) = \frac{1}{6} \neq P(A_1)P(A_2)$
$P(A_1 \cap A_3) = p(4) = \frac{1}{6} \neq P(A_1)P(A_3)$
Hence, $P(A_i \cap A_j) \neq P(A_i)P(A_j)$ for all $i \neq j$.

(iii) No.

Remark: This counter-example shows that one product condition does not imply independence.

Problem 4. In an experiment, A, B, C, and D are events with probabilities $P(A \cup B) = \frac{5}{8}$, $P(A) = \frac{3}{8}$, $P(C \cap D) = \frac{1}{3}$, and $P(C) = \frac{1}{2}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find

(i) $P(A \cap B)$

(ii) $P(B)$

(iii) $P(A \cap B^c)$

(iv) $P(A \cup B^c)$

(b) Are A and B independent?

(c) Find

(i) $P(D)$

(ii) $P(C \cap D^c)$

(iii) $P(C^c \cap D^c)$

(iv) $P(C|D)$

(v) $P(C \cup D)$

(vi) $P(C \cup D^c)$

(d) Are C and D^c independent?

Solution:

(a)

(i) Because $A \perp B$, we have $A \cap B = \emptyset$ and hence $P(A \cap B) = \boxed{0}$.

(ii) Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, $P(B) = P(A \cup B) - P(A) + P(A \cap B) = 5/8 - 3/8 + 0 = 2/8 = boxed 1/4$.

(iii) $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) = \boxed{3/8}$.

(iv) Start with $P(A \cup B^c) = 1 - P(A^c \cap B)$. Now, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) = 1/4$. Hence, $P(A \cup B^c) = 1 - 1/4 = \boxed{3/4}$.

(b) Events A and B are not independent because $P(A \cap B) \neq P(A)P(B)$.

(c)

(i) Because $C \perp\!\!\!\perp D$, we have $P(C \cap D) = P(C)P(D)$. Hence, $P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1/3}{1/2} = \boxed{2/3}$.

(ii) $P(C \cap D^c) = P(C) - P(C \cap D) = 1/2 - 1/3 = \boxed{1/6}$.
Alternatively, because $C \perp\!\!\!\perp D$, we know that $C \perp\!\!\!\perp D^c$. Hence, $P(C \cap D^c) = P(C)P(D^c) = \frac{1}{2}\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iii) First, we find $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = 5/6$. Hence, $P(C^c \cap D^c) = 1 - P(C \cup D) = 1 - 5/6 = \boxed{1/6}$.
Alternatively, because $C \perp\!\!\!\perp D$, we know that $C^c \perp\!\!\!\perp D^c$. Hence, $P(C^c \cap D^c) = P(C^c)P(D^c) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iv) Because $C \perp\!\!\!\perp D$, we have $P(C|D) = P(C) = \boxed{1/2}$.

(v) In part (iii), we already found $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = \boxed{5/6}$.

(vi) $P(C \cup D^c) = 1 - P(C^c \cap D) = 1 - P(C^c)P(D) = 1 - \frac{1}{2}\frac{2}{3} = \boxed{2/3}$. Note that we use the fact that $C^c \perp\!\!\!\perp D$ to get the second equality.
Alternatively, $P(C \cup D^c) = P(C) + P(D^c) - P(C \cap D^C)$. From (i), we have $P(D) = 2/3$. Hence, $P(D^c) = 1 - 2/3 = 1/3$. From (ii), we have $P(C \cap D^C) = 1/6$. Therefore, $P(C \cup D^c) = 1/2 + 1/3 - 1/6 = 2/3$.

(d) Yes. We know that if $C \perp\!\!\!\perp D$, then $C \perp\!\!\!\perp D^c$.

Extra Questions

Here are optional questions for those who want more practice.

Problem 5. Show that if A and B are independent events, then so are A and B^c, A^c and B, and A^c and B^c.

Solution: To show that two events C_1 and C_2 are independent, we need to show that $P(C_1 \cap C_2) = P(C_1)P(C_2)$.

(a) Note that
$$P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B).$$
Because $A \perp\!\!\!\perp B$, the last term can be factored in to $P(A)P(B)$ and hence
$$P(A \cap B^c) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$$

(b) By interchanging the role of A and B in the previous part, we have
$$P(A^c \cap B) = P(B \cap A^c) = P(B)\,P(A^c).$$

(c) From set theory, we know that $A^c \cap B^c = (A \cup B)^c$. Therefore,
$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B).$$
Because $A \perp\!\!\!\perp B$, we have
$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c). \end{aligned}$$

Remark: By interchanging the roles of A and A^c and/or B and B^c, it follows that if any one of the four pairs is independent, then so are the other three. [Gubner, 2006, p.31]

Problem 6. Anne and Betty go fishing. Find the conditional probability that Anne catches no fish given that at least one of them catches no fish. Assume they catch fish independently and that each has probability $0 < p < 1$ of catching no fish. [Gubner, 2006, Q2.62]

Hint: Let A be the event that Anne catches no fish and B be the event that Betty catches no fish. Observe that the question asks you to evaluate $P(A|(A \cup B))$.

Solution: From the question, we know that A and B are independent. The event "at least one of the two women catches nothing" can be represented by $A \cup B$. So we have

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B) - P(A)P(B)} = \frac{p}{2p - p^2} = \boxed{\frac{1}{2 - p}}.$$

Problem 7. Show that

(a) $P(A \cap B \cap C) = P(A) \times P(B|A) \times P(C|A \cap B)$.

(b) $P(B \cap C|A) = P(B|A)P(C|B \cap A)$

Solution:

(a) We can see directly from the definition of $P(B|A)$ that

$$P(A \cap B) = P(A)P(B|A).$$

Similarly, when we consider event $A \cap B$ and event C, we have

$$P(A \cap B \cap C) = P(A \cap B)\, P(C|A \cap B).$$

Combining the two equalities above, we have

$$P(A \cap B \cap C) = P(A) \times P(B|A) \times P(C|A \cap B).$$

(b) By definition,

$$P(B \cap C|A) = \frac{P(A \cap B \cap C)}{P(A)}.$$

Substitute $P(A \cap B \cap C)$ from part (a) to get

$$P(B \cap C|A) = \frac{P(A) \times P(B|A) \times P(C|A \cap B)}{P(A)} = P(B|A) \times P(C|A \cap B).$$

Problem 8. An article in the British Medical Journal ["Comparison of Treatment of Renal Calculi by Operative Surgery, Percutaneous Nephrolithotomy, and Extracorporeal Shock Wave Lithotripsy" (1986, Vol. 82, pp. 879892)] provided the following discussion of success rates in kidney stone removals. Open surgery (OS) had a success rate of 78% (273/350) while a newer method, percutaneous nephrolithotomy (PN), had a success rate of 83% (289/350). This newer method looked better, but the results changed when stone diameter was considered. For stones with diameters less than two centimeters, 93% (81/87) of cases of open surgery were successful compared with only 87% (234/270) of cases of PN. For stones greater than or equal to two centimeters, the success rates were 73% (192/263) and 69% (55/80) for open surgery and PN, respectively. Open surgery is better for both stone sizes, but less successful in total. In 1951, E. H. Simpson pointed out this apparent contradiction (known as Simpson's Paradox) but the hazard still persists today. Explain how open surgery can be better for both stone sizes but worse in total. [Montgomery and Runger, 2010, Q2-115]

Solution: First, let's recall the total probability theorem:

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A|B)\,P(B) + P(A|B^c)\,P(B^c). \end{aligned}$$

We can see that $P(A)$ does not depend only on $P(A \cap B)$ and $P(A|B^c)$. It also depends on $P(B)$ and $P(B^c)$. In the extreme case, we may imagine the case with $P(B) = 1$ in which $P(A) = P(A|B)$. At another extreme, we may imagine the case with $P(B) = 0$ in which $P(A) = P(A|B^c)$. Therefore, depending on the value of $P(B)$, the value of $P(A)$ can be anywhere between $P(A|B)$ and $P(A|B^c)$.

Now, let's consider events A_1, B_1, A_2, and B_2. Let $P(A_1|B_1) = 0.93$ and $P(A_1|B_1^c) = 0.73$. Therefore, $P(A_1) \in [0.73, 0.93]$. On the other hand, let $P(A_2|B_2) = 0.87$ and $P(A_2|B_2^c) = 0.69$. Therefore, $P(A_2) \in [0.69, 0.87]$. With small value of $P(B_1)$, the value of $P(A_1)$ can be 0.78 which is closer to its lower end of the bound. With large value of $P(B_2)$, the value of $P(A_2)$ can be 0.83 which is closer to its upper end of the bound. Therefore, even though $P(A_1|B_1) > P(A_2|B_2) = 0.87$ and $P(A_1|B_1^c) > P(A_2|B_2^c)$, it is possible that $P(A_1) < P(A_2)$.

In the context of the paradox under consideration, note that the success rate of PN with small stones (87%) is higher than the success rate of OS with large stones (73%). Therefore, by having a lot of large stone cases to be tested under OS and also have a lot of small stone cases to be tested under PN, we can create a situation where the overall success rate of PN comes out to be better then the success rate of OS. This is exactly what happened in the study as shown in Table 4.1.

Open surgery

	success	failure	sample size	sample percentage	conditional success rate
large stone	192	71	263	75%	73%
small stone	81	6	87	25%	93%
overall summary	273	77	350	100%	78%

PN

	success	failure	sample size	sample percentage	conditional success rate
large stone	55	25	80	23%	69%
small stone	234	36	270	77%	87%
overall summary	289	61	350	100%	83%

Table 4.1: Success rates in kidney stone removals.