ECS 315: Probability and Random Processes **2013/1**

HW Solution 3 — Due: July 12

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Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)
The extra question at the end is optional.

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1.

(a) Suppose that $P(A|B) = 0.4$ and $P(B) = 0.5$ Determine the following:

(i) $P(A \cap B)$

(ii) $P(A^c \cap B)$

[Montgomery and Runger, 2010, Q2-105]

(b) Suppose that $P(A|B) = 0.2$, $P(A|B^c) = 0.3$ and $P(B) = 0.8$ What is $P(A)$? [Montgomery and Runger, 2010, Q2-106]

Solution:

(a) Recall that $P(A \cap B) = P(A|B)P(B)$. Therefore,

(i) $P(A \cap B) = 0.4 \times 0.5 = \boxed{0.2.}$

(ii) $P(A^c \cap B) = P(B \setminus A) = P(B) - P(A \cap B) = 0.5 - 0.2 = \boxed{0.3.}$
Alternatively, $P(A^c \cap B) = P(A^c|B)P(B) = (1 - P(A|B))P(B) = (1 - 0.4) \times 0.5 = 0.3$.

(b) By the total probability formula, $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0.2 \times 0.8 + 0.3 \times (1 - 0.8) = \boxed{0.22.}$

Problem 2. Due to an Internet configuration error, packets sent from New York to Los Angeles are routed through El Paso, Texas with probability 3/4. Given that a packet is routed through El Paso, suppose it has conditional probability 1/3 of being dropped. Given that a packet is not routed through El Paso, suppose it has conditional probability 1/4 of being dropped.

(a) Find the probability that a packet is dropped.
Hint: Use total probability theorem.

(b) Find the conditional probability that a packet is routed through El Paso given that it is not dropped.
Hint: Use Bayes' theorem.

[Gubner, 2006, Ex.1.20]

Solution: To solve this problem, we use the notation $E = \{\text{routed through El Paso}\}$ and $D = \{\text{packet is dropped}\}$. With this notation, it is easy to interpret the problem as telling us that

$$P(D|E) = 1/3, \quad P(D|E^c) = 1/4, \quad \text{and } P(E) = 3/4.$$

(a) By the law of total probability,

$$\begin{aligned} P(D) &= P(D|E)P(E) + P(D|E^c)P(E^c) = (1/3)(3/4) + (1/4)(1 - 3/4) \\ &= 1/4 + 1/16 = \boxed{5/16} = 0.3125. \end{aligned}$$

(b) $P(E|D^c) = \frac{P(E \cap D^c)}{P(D^c)} = \frac{P(D^c|E)P(E)}{P(D^c)} = \frac{(1-1/3)(3/4)}{1-5/16} = \boxed{\frac{8}{11}} \approx 0.7273.$

Problem 3. You have two coins, a fair one with probability of heads $\frac{1}{2}$ and an unfair one with probability of heads $\frac{1}{3}$, but otherwise identical. A coin is selected at random and tossed, falling heads up. How likely is it that it is the fair one? [Capinski and Zastawniak, 2003, Q7.28]

Solution: Let F, U, and H be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively.

Because the coin is selected at random, the probability $P(F)$ of selecting the fair coin is $P(F) = \frac{1}{2}$. For fair coin, the conditional probability $P(H|F)$ of heads is $\frac{1}{2}$ For the unfair coin, $P(U) = 1 - P(F) = \frac{1}{2}$ and $P(H|U) = \frac{1}{3}$.

By the Bayes' formula, the probability that the fair coin has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{3}} = \frac{1}{1 + \frac{2}{3}} = \boxed{\frac{3}{5}}.$$

Problem 4. You have three coins in your pocket, two fair ones but the third biased with probability of heads p and tails $1-p$. One coin selected at random drops to the floor, landing heads up. How likely is it that it is one of the fair coins? [Capinski and Zastawniak, 2003, Q7.29]

Solution: Let F, U, and H be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively. We are given that

$$P(F) = \frac{2}{3}, \quad P(U) = \frac{1}{3}, \quad P(H|F) = \frac{1}{2}, P(H|U) = p.$$

By the Bayes' formula, the probability that the fair coin has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2} \times \frac{2}{3}}{\frac{1}{2} \times \frac{2}{3} + p \times \frac{1}{3}} = \boxed{\frac{1}{1+p}}.$$

Problem 5. Someone has rolled a fair dice twice. You know that one of the rolls turned up a face value of six. What is the probability that the other roll turned up a six as well? [Tijms, 2007, Example 8.1, p. 244]

Hint: Not $\frac{1}{6}$.

Solution: Take as sample space the set $\{(i,j)|i,j = 1,\ldots,6\}$, where i and j denote the outcomes of the first and second rolls. A probability of 1/36 is assigned to each element of the sample space. The event of two sixes is given by $A = \{(6,6)\}$ and the event of at least one six is given by $B = (1,6),\ldots,(5,6),(6,6),(6,5),\ldots,(6,1)$. Applying the definition of conditional probability gives

$$P(A|B) = P(A \cap B)/P(B) = \frac{1/36}{11/36}.$$

Hence the desired probability is $\boxed{1/11}$.

Problem 6. Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 99% of the time.

(a) What is $P(-|H)$, the conditional probability that a person tests negative given that the person does have the HIV virus?

(b) What is $P(H|+)$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive?

Solution:

(a) Because the test is correct 99% of the time,

$$P(-|H) = P(+|H^c) = \boxed{0.01}.$$

(b) Using Bayes' formula, $P(H|+) = \frac{P(+|H)P(H)}{P(+)}$, where $P(+)$ can be evaluated by the total probability formula:

$$P(+) = P(+|H)P(H) + P(+|H^c)P(H^c) = 0.99 \times 0.0002 + 0.01 \times 0.9998.$$

Plugging this back into the Bayes' formula gives

$$P(H|+) = \frac{0.99 \times 0.0002}{0.99 \times 0.0002 + 0.01 \times 0.9998} \approx \boxed{0.0194}.$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 2%. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Extra Question

Here is an optional questions for those who want more practice.

Problem 7.

(a) Suppose that $P(A|B) = 1/3$ and $P(A|B^c) = 1/4$. Find the range of the possible values for $P(A)$.

(b) Suppose that C_1, C_2, and C_3 partition Ω. Furthermore, suppose we know that $P(A|C_1) = 1/3$, $P(A|C_2) = 1/4$ and $P(A|C_3) = 1/5$. Find the range of the possible values for $P(A)$.

Solution: First recall the total probability theorem: Suppose we have a collection of events $B_1, B_2, \ldots, B_n$ which partitions Ω. Then,

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \cdots P(A \cap B_n) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots P(A|B_n)P(B_n) \end{aligned}$$

(a) Note that B and B^c partition Ω. So, we can apply the total probability theorem:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)).$$

You may check that, by varying the value of $P(B)$ from 0 to 1, we can get the value of $P(A)$ to be any number in the range $\left[\frac{1}{4}, \frac{1}{3}\right]$. Technically, we can not use $P(B) = 0$ because that would make $P(A|B)$ not well-defined. Similarly, we can not use $P(B) = 1$ because that would mean $P(B^c) = 0$ and hence make $P(A|B^c)$ not well-defined. Therfore, the range of $P(A)$ is $\boxed{\left(\frac{1}{4}, \frac{1}{3}\right).}$

Note that larger value of $P(A)$ is not possible because

$$P(A) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)) < \frac{1}{3}P(B) + \frac{1}{3}(1 - P(B)) = \frac{1}{3}.$$

Similarly, smaller value of $P(A)$ is not possible because

$$P(A) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)) > \frac{1}{4}P(B) + \frac{1}{3}(1 - P(B)) = \frac{1}{4}.$$

(b) Again, we apply the total probability theorem:

$$\begin{aligned} P(A) &= P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + P(A|C_3)P(C_3) \\ &= \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3). \end{aligned}$$

Because C_1, C_2, and C_3 partition Ω, we know that $P(C_1) + P(C_2) + P(C_3) = 1$. Now,

$$P(A) = \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3) < \frac{1}{3}P(C_1) + \frac{1}{3}P(C_2) + \frac{1}{3}P(C_3) = \frac{1}{3}.$$

Similarly,

$$P(A) = \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3) > \frac{1}{5}P(C_1) + \frac{1}{5}P(C_2) + \frac{1}{5}P(C_3) = \frac{1}{5}.$$

Therefore, $P(A)$ must be inside $\left(\frac{1}{5}, \frac{1}{3}\right)$.

You may check that any value of $P(A)$ in the range $\boxed{\left(\frac{1}{5}, \frac{1}{3}\right)}$ can be obtained by first setting the value of $P(C_2)$ to be close to 0 and varying the value of $P(C_1)$ from 0 to 1.