**ECS 315: Probability and Random Processes** **2013/1**

# HW Solution 2 — Due: July 5

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## Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

**Problem 1.** If $A$, $B$, and $C$ are disjoint events with $P(A) = 0.2$, $P(B) = 0.3$ and $P(C) = 0.4$, determine the following probabilities:

(a) $P(A \cup B \cup C)$

(b) $P(A \cap B \cap C)$

(c) $P(A \cap B)$

(d) $P((A \cup B) \cap C)$

(e) $P(A^c \cap B^c \cap C^c)$

[Montgomery and Runger, 2010, Q2-75]

***Solution***:

(a) Because $A$, $B$, and $C$ are disjoint, $P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.3 + 0.2 + 0.4 = \boxed{0.9.}$

(b) Because $A$, $B$, and $C$ are disjoint, $A \cap B \cap C = \emptyset$ and hence $P(A \cap B \cap C) = P(\emptyset) = \boxed{0}$.

(c) Because $A$ and $B$ are disjoint, $A \cap B = \emptyset$ and hence $P(A \cap B) = P(\emptyset) = \boxed{0}$.

(d) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. By the disjointness among $A$, $B$, and $C$, we have $(A \cap C) \cup (B \cap C) = \emptyset \cup \emptyset = \emptyset$. Therefore, $P((A \cup B) \cap C) = P(\emptyset) = \boxed{0}$.

(e) From $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$, we have $P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C) = 1 - 0.9 = \boxed{0.1.}$

**Problem 2.** The sample space of a random experiment is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let $A$ denote the event $\{a, b, c\}$, and let $B$ denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

[Montgomery and Runger, 2010, Q2-55]

***Solution***:

(a) Recall that the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Therefore,

$$\begin{aligned} P(A) &= P(\{a, b, c\}) = P(\{a\}) + P(\{b\}) + P(\{c\}) \\ &= 0.1 + 0.1 + 0.2 = \boxed{0.4.} \end{aligned}$$

(b) Again, the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Thus,

$$\begin{aligned} P(B) &= P(\{c, d, e\}) = P(\{c\}) + P(\{d\}) + P(\{e\}) \\ &= 0.2 + 0.4 + 0.2 = \boxed{0.8.} \end{aligned}$$

(c) $P(A^c) = 1 - P(A) = 1 - 0.4 = \boxed{0.6.}$

(d) Note that $A \cup B = \Omega$. Hence, $P(A \cup B) = P(\Omega) = \boxed{1.}$

(e) $P(A \cap B) = P(\{c\}) = \boxed{0.2.}$

**Problem 3.**

(a) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Find the range of possible values for $P(A \cap B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.21]

(b) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Find the range of possible values for $P(A \cup B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.22]

***Solution***:

(a) We will first try to bound $P(A \cap B)$. Note that $A \cap B \subset A$ and $A \cap B \subset B$. Hence, we know that $P(A \cap B) \leq P(A)$ and $P(A \cap B) \leq P(B)$. To summarize, we now know that

$$P(A \cap B) \leq \min\{P(A), P(B)\}.$$

On the other hand, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Applying the fact that $P(A \cup B) \leq 1$, we then have

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

If the number of the RHS is $> 0$, then it is a new information. However, if the number on the RHS is negative, it is useless and we will use the fact that $P(A \cap B) \geq 0$. To summarize, we now know that

$$\max\{P(A) + P(B) - 1, 0\} \leq P(A \cap B).$$

In conclusion,

$$\max\{(P(A) + P(B) - 1), 0\} \leq P(A \cap B) \leq \min\{P(A), P(B)\}.$$

Plugging in the value $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$ gives the range $\boxed{\left[\frac{1}{6}, \frac{1}{2}\right]}$. The upper-bound can be obtained by constructing an example which has $A \subset B$. The lower-bound can be obtained by considering an example where $A \cup B = \Omega$.

(b) By monotonicity we must have

$$P(A \cup B) \geq \max\{P(A), P(B)\}.$$

On the other hand, we know that

$$P(A \cup B) \le P(A) + P(B).$$

If the RHS is $> 1$, then the inequality is useless and we simply use the fact that it must be $\le 1$. To summarize, we have

$$P(A \cup B) \le \min\{(P(A) + P(B)), 1\}.$$

In conclusion,

$$\max\{P(A), P(B)\} \le P(A \cup B) \le \min\{(P(A) + P(B)), 1\}.$$

Plugging in the value $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$, we have

$$P(A \cup B) \in \boxed{\left[\frac{1}{2}, \frac{5}{6}\right]}.$$

The upper-bound can be obtained by making $A \perp B$. The lower-bound is achieved when $B \subset A$.

**Problem 4.** Let $A$ and $B$ be events for which $P(A)$, $P(B)$, and $P(A \cup B)$ are known. Express the following probabilities in terms of the three known probabilities above.

(a) $P(A \cap B)$

(b) $P(A \cap B^c)$

(c) $P(B \cup (A \cap B^c))$

(d) $P(A^c \cap B^c)$

***Solution***:

(a) $P(A \cap B) = \boxed{P(A) + P(B) - P(A \cup B)}$. This property is shown in class.

(b) We have seen in class that $P(A \cap B^c) = P(A) - P(A \cap B)$. Plugging in the expression for $P(A \cap B)$ from the previous part, we have

$$P(A \cap B^c) = P(A) - (P(A) + P(B) - P(A \cup B)) = \boxed{P(A \cup B) - P(B)}.$$

Alternatively, we can start from scratch with the set identity $A \cup B = B \cup (A \cap B^c)$ whose union is a disjoint union. Hence,

$$P(A \cup B) = P(B) + P(A \cap B^c).$$

Moving $P(B)$ to the LHS finishes the proof.

(c) $P(B \cup (A \cap B^c)) = \boxed{P(A \cup B)}$ because $A \cup B = B \cup (A \cap B^c)$.

(d) $P(A^c \cap B^c) = \boxed{1 - P(A \cup B)}$ because $A^c \cap B^c = (A \cup B)^c$.