Y&G Q4.4.2

Wednesday, September 14, 2011 4:47 PM

(a) We need $\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = 1$

This is the shorthand for $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$.

$\mathbb{R} = (-\infty, \infty)$

$\mathbb{R}^2 = (-\infty,\infty) \times (-\infty,\infty)$

Cartesian product.

After we exclude the pairs (x,y) whose $f_{X,Y}(x,y) = 0$, we need

$$\underbrace{\int_0^1\int_0^1 c\,x y^2\,dx\,dy}_{} = 1$$

$$= c\int_0^1 x\,dx \int_0^1 y^2\,dy = c\left(\left.\frac{x^2}{2}\right|_0^1\right)\left(\left.\frac{y^3}{3}\right|_0^1\right)$$

$$= c \times \frac{1}{2} \times \frac{1}{3} = \frac{c}{6}$$

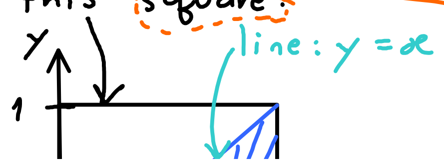
which means $c = 6$.

(b) Recall that $P[g(X,Y) \in B] = \iint_{R = \{(x,y):\, g(x,y) \in B\}} f_{X,Y}(x,y)\,dx\,dy$.

In particular, $P[X \geq Y] = \iint_{R = \{(x,y):\, x \geq y\}} f_{X,Y}(x,y)\,dx\,dy$.

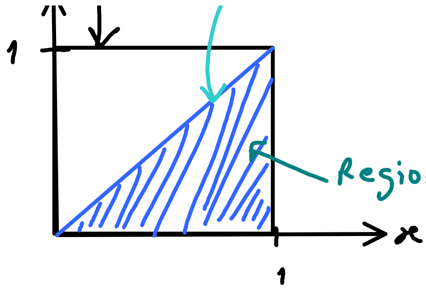
Therefore,

to find $P[X \geq Y]$, we first need to find the "region" R where $x \geq y$. Because $f_{X,Y}(x,y)$ is nonzero only in $(x,y) \in [0,1] \times [0,1]$, we only need to focus on (x,y) in this square.



We may call this square "the region of nonzero density".
In Y&G, this is called

"the region of nonzero density".
In Y&G, this is called
"the region of nonzero probability".



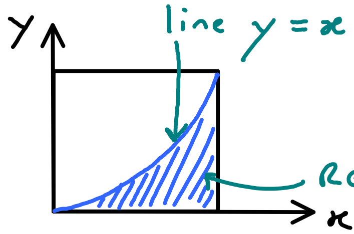
So, $P[X \geq Y] = \iint\limits_R f_{X,Y}(x,y)\,\underbrace{dy\,dx} = \int_0^1 \int_0^x c x y^2\, dy\, dx$

I integrate w.r.t. y first because it is easier to write the limits of the integrations.

$= c\int_0^1 x \int_0^x y^2 dy\, dx = c\int_0^1 x \left.\frac{y^3}{3}\right|_0^x dx$

$= c\int_0^1 \frac{x^4}{3} dx = c \left.\frac{x^5}{15}\right|_0^1 = c \times \frac{1}{15} = \frac{6}{15} = \frac{2}{5}.$

(c) The region R for $y < x^2$ is shown below



So, $P[Y < X^2] = \int_0^1 \int_0^{x^2} c x y^2 dy\, dx = c\int_0^1 x \int_0^{x^2} y^2 dy\, dx$

$= c\int_0^1 x \left.\frac{y^3}{3}\right|_0^{x^2} dx = \frac{c}{3}\int_0^1 x^7 dx = \frac{c}{3} \left.\frac{x^8}{8}\right|_0^1$

$= \frac{c}{24} = \frac{1}{4}.$

Y&G Q4.5.1

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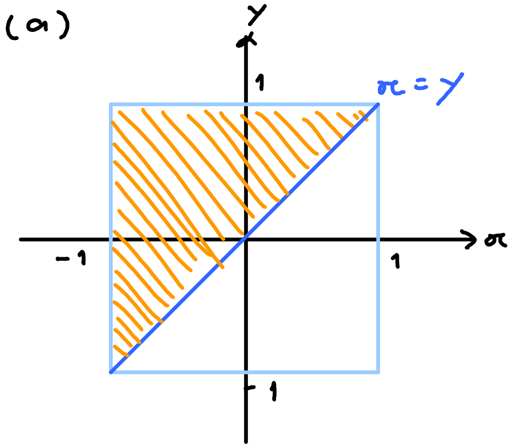
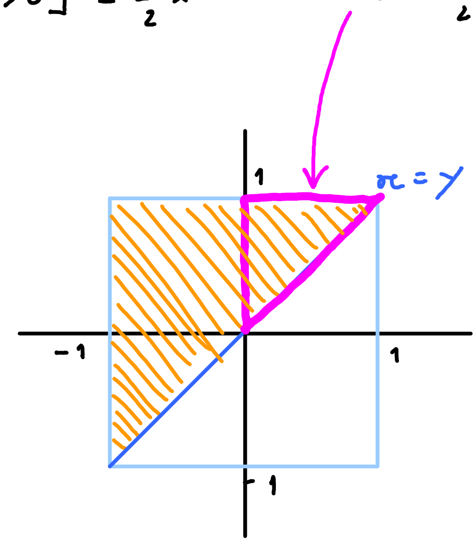
$$f_{X,Y}(x,y) = \begin{cases} 1/2, & -1 \le x \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a)



(b) $P[X>0] = \frac{1}{2} \times$ area here $= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.



(c) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy$

First, we observe that $f_X(x) = 0$ for $x < -1$ or $x > +1$ because $f_{X,Y}(x,y) = 0$ for any y.

For $-1 \le x \le 1$,

$$f_X(x) = \int_x^1 \frac{1}{2}\, dy = \frac{1}{2}(1-x)$$

Hence,

$$f_X(x) = \begin{cases} \frac{1}{2}(1-x), & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(d) $\mathbb{E}X = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{-1}^{1} x \frac{1}{2}(1-x)\, dx = -\frac{1}{3}$.

Y&G Q4.6.8

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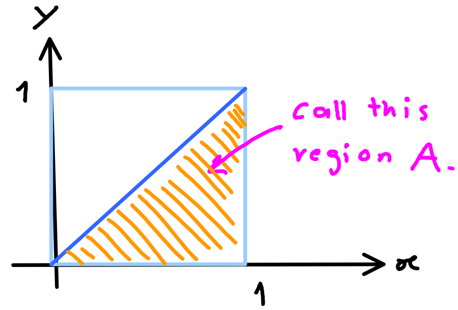
$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$



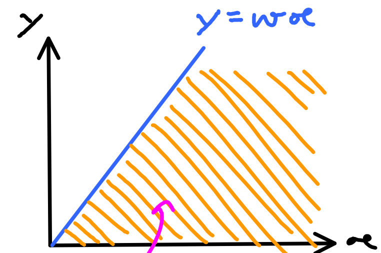
Note that both X and Y are nonnegative.

Hence, $W = \frac{Y}{X}$ is also nonnegative; that is $F_W(w) = 0$ for $w < 0$.

For $w \ge 0$, we have

$$F_W(w) = P[W \le w] = P\left[\frac{Y}{X} \le w\right] = P[Y \le wX]$$

Integrate $f_{X,Y}(x,y)$ over



Because (X,Y) is uniform over the region A,

$$F_W(w) = \underbrace{f_{X,Y}(x,y)}_{=2} \times (\text{area of } A \cap R)$$

Note that when $w > 1$, $A \subset R$ which implies area of $A \cap R$

= area of A

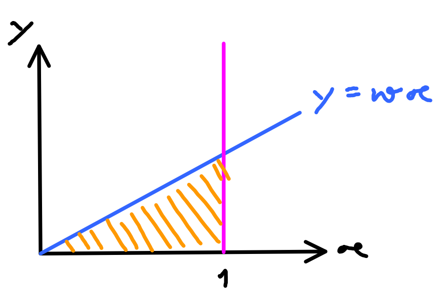
$= \frac{1}{2}$

So, $F_W(w) = 2 \times \frac{1}{2} = 1$,

when $0 \le w \le 1$, $A \cap R$ is the region shown below

with area $= \frac{1}{2} \times w \times 1 = \frac{w}{2}$.

So, $F_W(w) = 2 \times \frac{w}{2} = w$.



Hence,

$$F_W(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \le w \le 1 \\ 1, & w > 1. \end{cases}$$

From $f_W(w) = \frac{d}{dw} F_W(w)$, we have

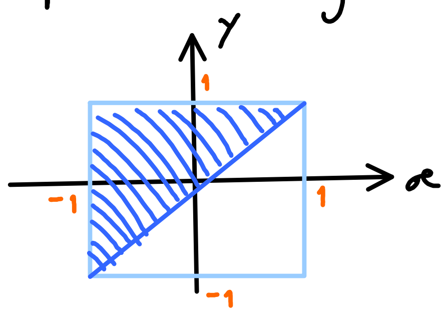
$$f_W(w) = \begin{cases} 1, & 0 \le w \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

So, W is uniform on $[0,1]$ and hence $\mathbb{E}W = \frac{0+1}{2} = \frac{1}{2}$.

Y&G Q4.7.12

Sunday, September 29, 2013 7:47 PM

First, we plot the region of possible pairs (x,y):



(a) $\mathbb{E}[XY] = \iint xy\, f_{X,Y}(x,y)\, dx\, dy = \int_{-1}^{1}\int_{x}^{1} xy \frac{1}{2}\, dy\, dx = 0.$

(b) $\mathbb{E}\left[e^{X+Y}\right] = \int_{-1}^{1}\int_{x}^{1} e^{x+y} \frac{1}{2}\, dy\, dx = \frac{e^{-2}}{4} + \frac{e^{2}}{4} - \frac{1}{2}.$

Y&G Q4.7.8

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(a) $$f_X(x) = \begin{cases} \int_0^2 \frac{x+y}{3}\,dy, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{2}{3}(x+1), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}X = \int_0^1 x \, \frac{2}{3}(x+1)\,dx = \frac{5}{9} \qquad \mathbb{E}X^2 = \int_0^1 x^2 \, \frac{2}{3}(x+1)\,dx = \frac{7}{18}$$

$$\text{Var}\, X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{13}{162} \approx 0.0802$$

(b) $\mathbb{E}Y = \frac{11}{9}$, $\quad \text{Var}\, Y = 23/81$

(c) $\text{Cov}[X,Y] = -1/81$

(d) $\mathbb{E}[X+Y] = \mathbb{E}X + \mathbb{E}Y = \frac{16}{9}$

(e) $\text{Var}[X+Y] = \text{Var}\, X + \text{Var}\, Y + 2\,\text{Cov}[X,Y] = \frac{13}{162} + \frac{46}{162} + 2\,\frac{(-2)}{162} = \frac{55}{162}$

Y&G Q4.11.1

Wednesday, September 28, 2011 9:53 AM

(a) Fact: $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$

It's best to start with an identity that we have already remember.
You may recall that the pdf of a Gaussian RV is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}.$$

We know that any pdf integrates to 1. Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx = 1.$$

In particular, when $m=0$,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} dx = 1$$

or, equivalently,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} dx = \sqrt{2\pi}\,\sigma.$$

Now, to get our stated fact above, we set $\frac{1}{2\sigma^2} = \alpha$.
This implies $\sigma = \frac{1}{\sqrt{2\alpha}}$ which gives

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{2\pi} \times \frac{1}{\sqrt{2\alpha}} = \sqrt{\frac{\pi}{\alpha}}.$$

Therefore, $\int_{-\infty}^{\infty} e^{-x^2/8} dx = \sqrt{8\pi} = 2\sqrt{2\pi}$ and $\int_{-\infty}^{\infty} e^{-y^2/18} dy = \sqrt{18\pi} = 3\sqrt{2\pi}$

Hence, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = c \times 2\sqrt{2\pi} \times 3\sqrt{2\pi} = 12\pi c$

This needs to be 1. Thus, $c = \frac{1}{12\pi}$.

Alternatively, note that $f_{X,Y}(x,y)$ is in the form of a bivariate Gaussian r.v. with $\rho = 0$ because there is no $x \cdot y$ term.

So, $\sigma_X = \sqrt{4} = 2$

$\sigma_Y = \sqrt{9} = 3$

and $c = \frac{1}{2\pi\sigma_X\sigma_Y} = \frac{1}{2\pi \times 2 \times 3} = \frac{1}{12\pi}$

(b) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \frac{c}{3\sqrt{2\pi}} e^{-x^2/8}$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \frac{c}{2\sqrt{2\pi}} e^{-y^2/18}$

Hence, $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ which implies $X \perp\!\!\!\perp Y$

Y&G Q4.4.3

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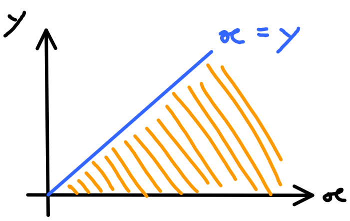
$$f_{X,Y}(x,y) = \begin{cases} 6e^{-2x}e^{-3y}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(Remark: Can you see right away that $X \sim \mathcal{E}(2)$, $Y \sim \mathcal{E}(3)$, and $X \perp\!\!\!\perp Y$?)



$$P[X > Y] = \int_0^\infty \int_0^x f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^\infty 2e^{-2x} \underbrace{\int_0^x 3e^{-3y}\,dy}_{P[Y < x] = 1 - P[Y \geq x]}$$

Recall that for $Y \sim \mathcal{E}(\lambda)$, $P[Y > y] = P[Y \geq y] = e^{-\lambda y}$

$$= \int_0^\infty 2e^{-2x}\left(1 - e^{-3x}\right)dx$$

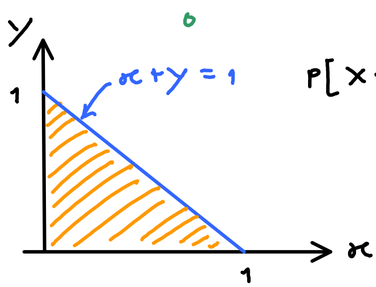
We can use the fact that pdf of exponential r.v. integrate to 1 to help us integrate other functions.

$$= \int_0^\infty 2e^{-2x}\,dx - \int_0^\infty 2e^{-5x}\,dx$$

$$= 1 - \frac{2}{5} = \boxed{\frac{3}{5}}$$

From $\int_0^\infty \lambda e^{-\lambda x}\,dx = 1$,

we have $\int_0^\infty e^{-\alpha x}\,dx = \frac{1}{\alpha}$



$$P[X + Y \leq 1] = \int_0^1 \int_0^{1-x} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^1 6e^{-2x} \int_0^{1-x} e^{-3y}\,dy\,dx$$

$$= \boxed{1 - 3e^{-2} + 2e^{-3}}$$