ECS 315: Probability and Random Processes **2013/1**

HW Solution 11 — Due: Sep 27

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Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

(e) For problems that use `MATLAB`, you must also print and submit your m-file along with the numerical results. In the beginning part of your code, there should be one commented line that indicates your name and student ID.

Problem 1. A webpage server can handle r requests per day. Find the probability that the server gets more than r requests at least once in n days. Assume that the number of requests on day i is $X_i \sim \mathcal{P}(\alpha)$ and that $X_1, \dots, X_n$ are independent.

Solution: [Gubner, 2006, Ex 2.10]

$$P\left[\bigcup_{i=1}^{n}[X_i > r]\right] = 1 - P\left[\bigcap_{i=1}^{n}[X_i \le r]\right] = 1 - \prod_{i=1}^{n} P\left[X_i \le r\right]$$
$$= 1 - \prod_{i=1}^{n}\left(\sum_{k=0}^{r}\frac{\alpha^k e^{-\alpha}}{k!}\right) = \boxed{1 - \left(\sum_{k=0}^{r}\frac{\alpha^k e^{-\alpha}}{k!}\right)^n}.$$

Problem 2. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

(a) Use `MATLAB` to evaluate the following quantities:

(i) $\mathbb{E}[XY]$

x \ y	2	4	5
1	0.02	0.10	0.08
3	0.08	0.32	0.40

(ii) $\mathbb{E}[(X-3)(Y-2)]$

(iii) $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)]$

(iv) $\text{Cov}[X,Y]$

(v) $\rho_{X,Y}$

(b) Find $\rho_{X,X}$

(c) Calculate the following quantities using the values of $\text{Var}\,X$, $\text{Cov}[X,Y]$, and $\rho_{X,Y}$ that you got earlier.

(i) $\text{Cov}[3X+4, 6Y-7]$

(ii) $\rho_{3X+4,6Y-7}$

(iii) $\text{Cov}[X, 6X-7]$

(iv) $\rho_{X,6X-7}$

Solution: The `MATLAB` codes are provided in the file `P_XY_EVarCov.m`.

(a)

(i) From `MATLAB`, $\mathbb{E}[XY] = 11.16$

(ii) From `MATLAB`, $\mathbb{E}[(X-3)(Y-2)] = -0.88$

(iii) From `MATLAB`, $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)] = 104$

(iv) From `MATLAB`, $\text{Cov}[X,Y] = 0.032$

(v) From `MATLAB`, $\rho_{X,Y} = 0.044677$

(b) $\rho_{X,X} = \frac{\text{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\text{Var}[X]}{\sigma_X^2} = 1$

(c)

(i) $\text{Cov}[3X+4, 6Y-7] = 3 \times 6 \times \text{Cov}[X,Y] \approx 3 \times 6 \times 0.032 \approx \boxed{0.576}$.

(ii) Note that

$$\rho_{aX+b,cY+d} = \frac{\text{Cov}\left[aX+b, cY+d\right]}{\sigma_{aX+b}\sigma_{cY+d}} = \frac{ac\text{Cov}\left[X,Y\right]}{|a|\sigma_X |c|\sigma_Y} = \frac{ac}{|ac|}\rho_{X,Y} = \text{sign}(ac) \times \rho_{X,Y}.$$

Hence, $\rho_{3X+4,6Y-7} = \text{sign}(3 \times 4)\rho_{X,Y} = \rho_{X,Y} = \boxed{0.0447}$.

(iii) $\text{Cov}\left[X, 6X-7\right] = 1 \times 6 \times \text{Cov}\left[X, X\right] = 6 \times \text{Var}[X] \approx \boxed{3.84}$.

(iv) $\rho_{X,6X-7} = \text{sign}(1 \times 6) \times \rho_{X,X} = \boxed{1}$.

Problem 3. Suppose $X \sim$ binomial$(5, 1/3)$, $Y \sim$ binomial$(7, 4/5)$, and $X \perp\!\!\!\perp Y$. Evaluate the following quantities.

(a) $\mathbb{E}\left[(X-3)(Y-2)\right]$

(b) $\mathbb{E}\left[X(Y^3 - 11Y^2 + 38Y)\right]$

(c) $\text{Cov}\left[X, Y\right]$

(d) $\rho_{X,Y}$

Solution: The `MATLAB` codes are provided in the file `P_XY_jointfromMarginal_indp_ECov.m`.

(a) $\mathbb{E}\left[(X-3)(Y-2)\right] = \boxed{-4.8}$

(b) $\mathbb{E}\left[X(Y^3 - 11Y^2 + 38Y)\right] = \boxed{82.13}$

(c) $\text{Cov}\left[X, Y\right] = \boxed{0}$ because $X \perp\!\!\!\perp Y$. (MATLAB should give a very small number.)

(d) $\rho_{X,Y} = \boxed{0}$ because $\text{Cov}\left[X, Y\right] = 0$ (MATLAB should give a very small number.)

Problem 4. Suppose $\text{Var}\, X = 5$. Find $\text{Cov}\left[X, X\right]$ and $\rho_{X,X}$.

Solution:

(a) $\text{Cov}\left[X, X\right] = \mathbb{E}\left[(X - \mathbb{E}X)(X - \mathbb{E}X)\right] = \mathbb{E}\left[(X - \mathbb{E}X)^2\right] = \text{Var}\, X = \boxed{5}$.

(b) $\rho_{X,X} = \frac{\text{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\text{Var}\, X}{\sigma_X^2} = \frac{\text{Var}\, X}{\text{Var}\, X} = \boxed{1}$.

Problem 5. Suppose we know that $\sigma_X = \frac{\sqrt{21}}{10}$, $\sigma_Y = \frac{4\sqrt{6}}{5}$, $\rho_{X,Y} = -\frac{1}{\sqrt{126}}$.

(a) Find $\mathrm{Var}[X+Y]$.

(b) Find $\mathbb{E}\left[(Y-3X+5)^2\right]$. Assume $\mathbb{E}\left[Y-3X+5\right]=1$.

Solution:

(a) First, we know that $\mathrm{Var}\,X = \sigma_X^2 = \frac{21}{100}$, $\mathrm{Var}\,Y = \sigma_Y^2 = \frac{96}{25}$, and $\mathrm{Cov}\left[X,Y\right] = \rho_{X,Y} \times \sigma_X \times \sigma_Y = -\frac{2}{25}$. Now,

$$\begin{aligned}\mathrm{Var}\left[X+Y\right] &= \mathbb{E}\left[\left((X+Y)-\mathbb{E}\left[X+Y\right]\right)^2\right] = \mathbb{E}\left[\left((X-\mathbb{E}X)+(Y-\mathbb{E}Y)\right)^2\right] \\ &= \mathbb{E}\left[(X-\mathbb{E}X)^2\right] + 2\mathbb{E}\left[(X-\mathbb{E}X)(Y-\mathbb{E}Y)\right] + \mathbb{E}\left[(Y-\mathbb{E}Y)^2\right] \\ &= \mathrm{Var}\,X + 2\mathrm{Cov}\left[X,Y\right] + \mathrm{Var}\,Y \\ &= \boxed{\frac{389}{100}} = 3.89.\end{aligned}$$

Remark: It is useful to remember that

$$\mathrm{Var}\left[X+Y\right] = \mathrm{Var}\,X + 2\mathrm{Cov}\left[X,Y\right] + \mathrm{Var}\,Y.$$

Note that when X and Y are uncorrelated, $\mathrm{Var}\left[X+Y\right] = \mathrm{Var}\,X + \mathrm{Var}\,Y$. This simpler formula also holds when X and Y are independence because independence is a stronger condition.

(b) First, we write

$$Y - aX - b = (Y-\mathbb{E}Y) - a(X-\mathbb{E}X) - \underbrace{(a\mathbb{E}X + b - \mathbb{E}Y)}_{c}.$$

Now, using the expansion

$$(u+v+t)^2 = u^2 + v^2 + t^2 + 2uv + 2ut + 2vt,$$

we have

$$\begin{aligned}(Y-aX-b)^2 = {} & (Y-\mathbb{E}Y)^2 + a^2(X-\mathbb{E}X)^2 + c^2 \\ & - 2a(X-\mathbb{E}X)(Y-\mathbb{E}Y) - 2c(Y-\mathbb{E}Y) + 2a(X-\mathbb{E}X)c.\end{aligned}$$

Recall that $\mathbb{E}\left[X-\mathbb{E}X\right] = \mathbb{E}\left[Y-\mathbb{E}Y\right] = 0$. Therefore,

$$\mathbb{E}\left[(Y-aX-b)^2\right] = \mathrm{Var}\,Y + a^2\,\mathrm{Var}\,X + c^2 - 2a\mathrm{Cov}\left[X,Y\right]$$

Plugging back the value of c, we have

$$\boxed{\mathbb{E}\left[(Y - aX - b)^2\right] = \operatorname{Var} Y + a^2 \operatorname{Var} X + \left(\mathbb{E}\left[(Y - aX - b)\right]\right)^2 - 2a\operatorname{Cov}\left[X, Y\right]}.$$

Here, $a = 3$ and $b = -5$. Plugging these values along with the given quantities into the formula gives

$$\mathbb{E}\left[(Y - aX - b)^2\right] = \boxed{\frac{721}{100}} = 7.21.$$