ECS 315: Probability and Random Processes	**2013/1**

HW Solution 10 — Due: Sep 20

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. In wireless communications systems, fading is sometimes modeled by ***lognormal*** random variables. We say that a positive random variable Y is lognormal if $\ln Y$ is a normal random variable (say, with expected value m and variance σ^2). Find the pdf of Y.

Hint: First, recall that the ln is the natural log function (log base e). Let $X = \ln Y$. Then we know that $X \sim \mathcal{N}(m, \sigma^2)$. Can you write Y as a function of X? In class, we talked about a formula that turns pdf of X to pdf of Y. Use it here.

Solution: Because $X = \ln(Y)$, we have $Y = e^X$. Exponential function gives positive number. So, we know that Y is nonnegative. In particular, $f_Y(y) = 0$ for $y < 0$

For $y > 0$, as discussed in class, a shortcut for finding the pdf of Y is to think about a small region of y's. The probability that Y will be in this region should be the same as the probability of the corresponding region of the x's:

$$f_X(x)|dx| = f_Y(y)|dy|. \tag{10.1}$$

Solving (10.1) for $f_Y(y)$, we have

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|. \tag{10.2}$$

Note that (10.1) and (10.2) hold only when the function $y = g(x)$ is strictly increasing or strictly decreasing. (Otherwise, these can be multiple roots (the x's) that produce the same y. Extension is possible; to cover multiple-root cases we modify the left side of (10.1) to be the sum of all small probabilities that X will be in the small neighborhood around each of the roots.

Because $x = \ln(y)$ for any $y > 0$, we have $\frac{dx}{dy} = \frac{1}{y}$. From (10.2), we then have

$$f_Y(y) = f_X(\ln(y)) \left|\frac{1}{y}\right| = \frac{1}{y} f_X(\ln(y)).$$

Here, $X \sim \mathcal{N}(m, \sigma^2)$. Therefore,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

and

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{1}{2}\left(\frac{ln(y)-m}{\sigma}\right)^2}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}}$$

Note that we also arbitrarily set $f_Y(0) = 0$.

Problem 2. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

x \ y	2	4	5
1	0.02	0.10	0.08
3	0.08	0.32	0.40

Use `MATLAB` or `Excel` to evaluate the following quantities:

(a) The marginal pmf $p_X(x)$

(b) The marginal pmf $p_Y(y)$

(c) $\mathbb{E}X$

(d) $\text{Var}\, X$

(e) $\mathbb{E}Y$

(f) $\text{Var}\, Y$

(g) $P[XY < 6]$

(h) $P[X = Y]$

Solution: The `MATLAB` codes are provided in the file `P_XY_marginal.m`.

(a) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 0.2, & x = 1 \\ 0.8, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

(b) The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 0.1, & y = 2 \\ 0.42, & y = 4 \\ 0.48, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

(c) From `MATLAB` or `Excel`, $\mathbb{E}X = 2.6$

(d) From `MATLAB` or `Excel`, $\operatorname{Var} X = 0.64$

(e) From `MATLAB` or `Excel`, $\mathbb{E}Y = 4.28$

(f) From `MATLAB` or `Excel`, $\operatorname{Var} Y = 0.8016$

(g) $P[XY < 6] = P[X = 1] = 0.2$

(h) $P[X = Y] = 0$

Problem 3. The input X and output Y of a system subject to random perturbations are described probabilistically by the joint pmf $p_{X,Y}(x, y)$, where $x = 1, 2, 3$ and $y = 1, 2, 3, 4, 5$. Let P denote the joint pmf matrix whose i,j entry is $p_{X,Y}(i, j)$, and suppose that

$$P = \frac{1}{71}\begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

(a) Find the marginal pmfs $p_X(x)$ and $p_Y(y)$.

(b) Find $\mathbb{E}X$

(c) Find $\mathbb{E}Y$

(d) Find $\operatorname{Var} X$

(e) Find $\operatorname{Var} Y$

Solution: All of the calculations in this question are simply plugging numbers into appropriate formula. The MATLAB codes are provided in the file `P_XY_marginal_2.m`.

(a) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 26/71, & x = 1 \\ 25/71, & x = 2 \\ 20/71, & x = 3 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.3662, & x = 1 \\ 0.3521, & x = 2 \\ 0.2817, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 13/71, & y = 1 \\ 8/71, & y = 2 \\ 21/71, & y = 3 \\ 15/71, & y = 4 \\ 14/71, & y = 5 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.1831, & y = 1 \\ 0.1127, & y = 2 \\ 0.2958, & y = 3 \\ 0.2113, & y = 4 \\ 0.1972, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

(b) $\mathbb{E}X = \frac{136}{71} \approx 1.9155$

(c) $\mathbb{E}Y = \frac{222}{71} \approx 3.1268$

(d) $\operatorname{Var} X = \frac{3230}{5041} \approx 0.6407$

(e) $\operatorname{Var} Y = \frac{9220}{5041} \approx 1.8290$

Problem 4. Suppose $X \sim$ binomial$(5, 1/3)$, $Y \sim$ binomial$(7, 4/5)$, and $X \perp\!\!\!\perp Y$.

(a) A vector describing the pmf of X can be created by the MATLAB expression:

```
x = 0:5; pX = binopdf(x,5,1/3).
```

What is the expression that would give `pY`, a corresponding vector describing the pmf of Y?

(b) Use `pX` and `pY` from part (a), how can you create the joint pmf matrix in MATLAB? Do not use "for-loop", "while-loop", "if statement". Hint: Multiply them in an appropriate orientation.

(c) Use MATLAB to evaluate the following quantities. Again, do not use "for-loop", "while-loop", "if statement".

(i) $\mathbb{E}X$

(ii) $P[X = Y]$

(iii) $P[XY < 6]$

Solution: The MATLAB codes are provided in the file `P_XY_jointfromMarginal_indp.m`.

(a) `y = 0:7; pY = binopdf(y,7,4/5);`

(b) `P = pX.'*pY;`

(c)

(i) $\mathbb{E}X = \boxed{1.667}$

(ii) $P[X = Y] = \boxed{0.0121}$

(iii) $P[XY < 6] = \boxed{0.2727}$