Problem 1. (25pt) Random variables X and Y have the following joint pmf

$$p_{X,Y}(x,y) = \begin{cases} c(x+y), & x \in \{1,3\} \text{ and } y \in \{2,4\}, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (1pt) Check that $c = 1/20$.

(b) (2pt) Find $P\left[X^2 + Y^2 = 13\right]$.

(c) (2pt) Find $p_X(x)$.

(d) (2pt) Find $\mathbb{E}X$.

(e) (2pt) Find $p_{Y|X}(y|1)$. Note that your answer should be of the form

$$p_{Y|X}(y|1) = \begin{cases} ?, & y = 2, \\ ?, & y = 4, \\ 0, & \text{otherwise.} \end{cases}$$

(f) (2pt) Find $\mathbb{E}[Y|X = 1]$.

(g) (2pt) Find $p_{Y|X}(y|3)$.

(h) (2pt) Find $\mathbb{E}[Y|X = 3]$.

(i) (2pt) **Use** iterated expectation to find $\mathbb{E}Y$ from your answers in parts (c), (e) and (g).

(j) (2pt) Find $\mathbb{E}[XY]$.

(k) (2pt) Check that $\operatorname{Cov}[X,Y] = -\frac{1}{25}$.

(l) Let $Z = X + Y$.

(i) (2pt) Find the pmf of Z.

(ii) (2pt) Find $\mathbb{E}Z$.

Problem 2. (14pt) Suppose X is uniformly distributed on the interval $(1,2)$. ($X \sim \mathcal{U}(1,2)$.) Let $Y = \frac{1}{X^2}$.

(a) (2pt) Plot the pdf $f_X(x)$ of X.

(b) (2pt) Plot the cdf $F_X(x)$ of X.

(c) (2pt) Find $P[\sin(X) > 0]$. Assume that X is in radians.

(d) (5pt) Find $f_Y(y)$.

(e) (3pt) Find $\mathbb{E}Y$.

Problem 3. (16pt) Random variables X and Y have joint pdf

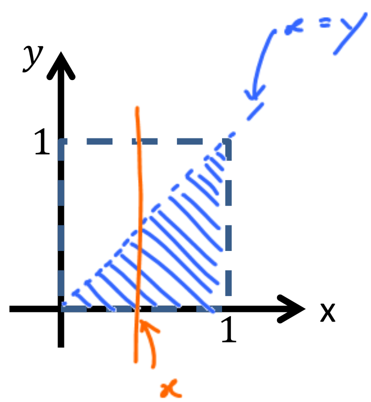
$$f_{X,Y}(x,y) = \begin{cases} c, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (1pt) Check that $c = 2$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1 \quad \Rightarrow \frac{c}{2} = 1 \Rightarrow 2$$

base area × height

(b) (1pt) In the picture below, specify the region of nonzero pdf.



(c) (2pt) Find the marginal density $f_X(x)$.

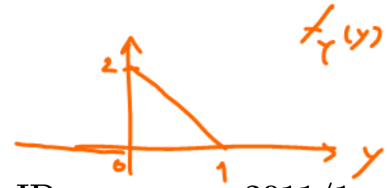
$$f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int 0\,dy = 0 \qquad x < 0 \text{ or } x > 1$$

$$= \int_0^x c\,dy = cx \qquad 0 \le x \le 1$$

(d) (2pt) Check that $\mathbb{E}X = \frac{2}{3}$.

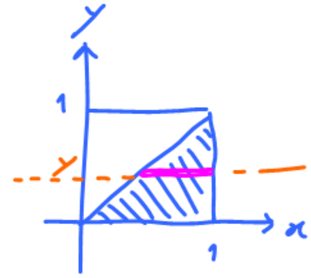
$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x(2x)\,dx = \int_0^1 2x^2\,dx = \left.\frac{2x^3}{3}\right|_0^1 = \frac{2}{3}$$



(e) (2pt) Find the marginal density $f_Y(y)$.

$c(1-y) = 2(1-y)$

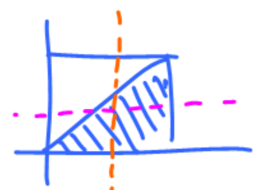
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_y^1 c\,dx, & 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$



(f) (2pt) Find $\mathbb{E}Y$

$$\mathbb{E}Y = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^1 y\,2(1-y)\,dy = 2\int_0^1 y - y^2\,dy = 2\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1$$

$$= \frac{2}{6} = \frac{1}{3}$$



(g) (2pt) Find $\mathbb{E}[XY]$

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dy\,dx = \int_0^1 \int_0^x xy\, c\;dy\,dx = \frac{1}{4}$$

$$= \int_0^1 \int_y^1 xy\, c\;dx\,dy$$

(h) (2pt) Are X and Y uncorrelated?

No

$$\mathbb{E}[XY] = \frac{1}{4} \neq \mathbb{E}X\,\mathbb{E}Y \Rightarrow \text{correlated.}$$

(i) (2pt) Are X and Y independent?

No. independent $\Rightarrow$ uncorrelated

Not uncorrelated $\Rightarrow$ not independent.

Problem 4. (6pt) Let X_1 and X_2 be i.i.d. $\mathcal{E}(1)$ (exponential pdf with rate parameter $\lambda = 1$).

(a) (2pt) Find $P[X_1 = X_2]$.

(b) (1pt) Find $P\left[X_1^2 + X_2^2 = 13\right]$.

(c) (1pt*) Define $Y = \min\{X_1, X_2\}$. (For example, when $X_1 = 6$ and $X_2 = 4$, we have $Y = 4$.) Describe the random variable Y. Does it belong to any known family of random variables? If so, what is/are its parameters?

(d) (1pt*) Define $Y = \min\{X_1, X_2\}$ and $Z = \max\{X_1, X_2\}$. Find $f_{Y,Z}(2,1)$.

(e) (1pt**) Define $Y = \min\{X_1, X_2\}$ and $Z = \max\{X_1, X_2\}$. Find $f_{Y,Z}(1,2)$.

Problem 5. (8pt) Consider i.i.d. random variables $X_1, X_2, \ldots, X_{10}$. Define the sample mean M by

$$M = \frac{1}{10}\sum_{k=1}^{10} X_k.$$

Let

$$V_1 = \frac{1}{10}\sum_{k=1}^{10} (X_k - \mathbb{E}[X_k])^2.$$

and

$$V_2 = \frac{1}{10}\sum_{k=1}^{10} (X_k - M)^2.$$

Suppose $\mathbb{E}[X_k] = 1$ and $\text{Var}[X_k] = 2$.

(a) (3pt*) Find the variance of the product: $\text{Var}[X_1 X_2]$.

(b) (1pt) Find $\mathbb{E}[M]$.

(c) (1pt) Find $\text{Var}[M]$.

(d) (2pt) Find $\mathbb{E}[V_1]$.

(e) (1pt**) Find $\mathbb{E}[V_2]$.

Problem 6. (9pt) Consider the function

$$g(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Suppose $Y = g(X)$, where $X \sim \mathcal{U}(-2, 2)$.

Remark: The function g operates like a **full-wave rectifier** in that if a positive input voltage X is applied, the output is $Y = X$, while if a negative input voltage X is applied, the output is $Y = -X$.

(a) (2pt) Find $\mathbb{E}Y$.

(b) (4pt) Plot the cdf of Y.

(c) (3pt) Find the pdf of Y

Problem 7. (11 pt) Suppose a random variable X has density

$$f_X(x) = \frac{1}{4}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} + c\delta(x-1).$$

(a) (3 pt) Find c.

(b) (2 pt) Find $P[X = 1]$

(c) (2 pt) Find $P[X \leq 0]$

(d) (2 pt) Find $\mathbb{E}[X]$

~~(c)~~ (2 pt) Find Var X

Problem 8. (7pt) Recall that the characteristic function $\varphi_X(v) = \mathbb{E}\left[e^{jvX}\right]$ can be used to find the moments $\mathbb{E}\left[X^k\right]$ of a random variable X. In particular,

$$\left.\frac{d^k}{dv^k}\varphi_X(v)\right|_{v=0} = j^k\mathbb{E}\left[X^k\right]. \tag{2.1}$$

In this problem, assume that $X \sim \mathcal{N}(m, \sigma^2)$. This means the characteristic function of X is given by

$$\varphi_X(v) = \mathbb{E}\left[e^{jvX}\right] = e^{jvm - \frac{1}{2}v^2\sigma^2}. \tag{2.2}$$

(a) (3pt) Use (2.1) and (2.2) to show that $\mathbb{E}X = m$.

$\frac{d}{dv}\varphi_X(v) = \mathbb{E}\left[jXe^{jvX}\right]$

$\varphi_X'(0) = j\mathbb{E}X$

(b) (3pt) Use (2.1) and (2.2) to show that $\mathbb{E}\left[X^2\right] = \sigma^2 + m^2$.

(c) (1pt*) Find $\mathbb{E}\left[X^3\right]$ when $m = 3$ and $\sigma = 2$.

Problem 9. (6pt) Kakashi and Gai are eternal rivals. Kakashi is a little stronger than Gai and hence for each time that they fight, the probability that Kakashi wins is 0.55. In a competition, they fight n times (where n is odd). We will assume that the results of the fights are independent. The one who wins more will win the competition.

(a) (3pt) Suppose $n = 3$, what is the probability that Kakashi wins the competition.

(b) (3pt) The stronger person (Kakashi) should win the competition if n is very large. (By the law of large numbers, the proportion of fights that Kakashi wins should be close to 55%.) However, because the results are random and n can not be very large, we can not guarantee that Kakashi will win. However, it may be good enough if the probability that Kakashi wins the competition is greater than 0.85.

We want to find the minimal value of n such that the probability that Kakashi wins the competition is greater than 0.85.

Let N be the number of fights that Kakashi wins among the n fights. Then, we need

$$P\left[N > \frac{n}{2}\right] \geq 0.85. \tag{2.3}$$

Use the central limit theorem and Table 3.1 or Table 3.2 from [Yates and Goodman] to **approximate the minimal value of** n such that (2.3) is satisfied.