Quiz 4 Solution

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10:33 AM

Suppose $p_{X,Y}(x,y) = \begin{cases} cx^2y, & x \in \{1,2\}, y \in \{1,2\}, \\ 0, & \text{otherwise.} \end{cases}$

① Find the joint pmf matrix $\mathbf{P}_{X,Y}$

First we identify the supports for X and Y.
Note that the given joint pmf is nonzero only when $x \in \{1,2\}$ and $y \in \{1,2\}$.

Therefore, we conclude that a support of X is $\{1,2\}$ and a support of Y is $\{1,2\}$.

We can then express the joint pmf using the joint pmf matrix:

$$\mathbf{P}_{X,Y} = \begin{array}{c|cc} x \backslash y & 1 & 2 \\ \hline 1 & c\times 1^2 \times 1 & c\times 1^2 \times 2 \\ 2 & c\times 2^2 \times 1 & c\times 2^2\times 2 \end{array} = \begin{array}{c|cc} x \backslash y & 1 & 2 \\ \hline 1 & c & 2c \\ 2 & 4c & 8c \end{array}.$$

To find the constant c, recall that $\sum_x \sum_y p_{X,Y}(x,y) = 1$.

So, we need

$$c + 2c + 4c + 8c = 1$$
$$15c = 1$$
$$c = \frac{1}{15}.$$

Here, we only need to focus on the $(x,y)$-pairs that are in the joint pmf matrix because other pairs have zero probability.

Plugging this value in the joint pmf matrix above, we get

$$\mathbf{P}_{X,Y} = \begin{array}{c|cc} x \backslash y & 1 & 2 \\ \hline 1 & 1/15 & 2/15 \\ 2 & 4/15 & 8/15 \end{array}.$$

② Find $\mathbb{E}X$

To find $\mathbb{E}X$, we need $p_X(x)$. This can be obtained from the joint pmf matrix by summing along each row.

This is the same as applying the formula $p_X(x) = \sum_y p_{X,Y}(x,y)$.

$$
P_{X,Y} = \begin{array}{c|cc} x\backslash y & 1 & 2 \\ 1 & 1/15 & 2/15 \\ 2 & 4/15 & 8/15 \end{array}
\quad \begin{array}{l} \xrightarrow{\text{sum along each row}} 3/15 = 1/5 \\ \longrightarrow 12/15 = 4/5 \end{array}
$$

Therefore, we have $p_X(x) = \begin{cases} 1/5, & x = 1, \\ 4/5, & x = 2, \\ 0, & \text{otherwise}. \end{cases}$

This gives $\mathbb{E}X = \sum_x x\, p_X(x) = 1 \times \frac{1}{5} + 2 \times \frac{4}{5} = \frac{9}{5}$.

③ Are $X$ and $Y$ independent?

To check whether $X \perp\!\!\!\perp Y$, we need to show that

$$p_{X,Y}(x,y) = p_X(x)\, p_Y(y) \quad \underbrace{\text{for any } x, y.}$$

This can be reduced to only the $x$ in a support of $X$ and $y$ in a support of $Y$ because we automatically get $0 = 0 \times 0$ outside the supports.

We've already found $p_{X,Y}(x,y)$ and $p_X(x)$. So, we still need $p_Y(y)$. This can be obtained from summing along each column of the joint pmf matrix:

$$
P_{X,Y} = \begin{array}{c|cc} x\backslash y & 1 & 2 \\ 1 & 1/15 & 2/15 \\ 2 & 4/15 & 8/15 \\ & \downarrow & \downarrow \\ & 5/15 & 10/15 \\ & \shortparallel & \shortparallel \\ & 1/3 & 2/3 \end{array}
\quad \Leftarrow \text{Sum along each column.}
$$

So, we have $p_Y(y) = \begin{cases} 1/3, & y = 1, \\ 2/3, & y = 2, \\ 0, & \text{otherwise}. \end{cases}$

Now, to check for independence, we need to check whether

$$p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$$

for $x \in \{1,2\}$ and $y \in \{1,2\}$.

A concise way to do this is to express $p_X(x)$ and $p_Y(y)$

as row vectors $p_X$ and $p_Y$, respectively. Then check whether

$$(p_X)^T p_Y = P_{X,Y}.$$

Here, $p_X = [1/5 \quad 4/5]$ and $p_Y = [\frac{1}{3} \quad \frac{2}{3}]$.

Therefore, $$(p_X)^T p_Y = \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1/15 & 2/15 \\ 4/15 & 8/15 \end{bmatrix}.$$

This is exactly the same as the joint pmf matrix $P_{X,Y}$ that we found in part 1. Therefore, $X \perp\!\!\!\perp Y$.

4 Find $\mathbb{E}[XY]$

Method 1: When $X \perp\!\!\!\perp Y$, we know that

$$\mathbb{E}[h(X)g(Y)] = \mathbb{E}[h(X)]\,\mathbb{E}[g(Y)],$$

for "any" functions $h$ and $g$.

In particular, $\mathbb{E}[XY] = (\mathbb{E}X)(\mathbb{E}Y) = \frac{9}{5} \times \frac{5}{3} = 3$.

$\mathbb{E}X = \frac{9}{5}$, $\mathbb{E}Y = \frac{1}{3} \times 1 + \frac{2}{3} \times 2 = \frac{5}{3}$

Method 2: We can also use our general formula (LOTUS):

$$\mathbb{E}[g(X,Y)] = \sum_x \sum_y g(x,y)\, p_{X,Y}(x,y).$$

This formula works regardless of whether $X \perp\!\!\!\perp Y$.

Here,

$$\mathbb{E}[XY] = \sum_x \sum_y xy\, p_{X,Y}(x,y).$$

Method 2.1:

For conciseness, we first find a matrix that contains the values of $xy$:

$$\begin{array}{c|cc} x \backslash y & 1 & 2 \\ \hline 1 & 1\times 1 & 1\times 2 \\ 2 & 2\times 1 & 2\times 2 \end{array} = \begin{array}{c|cc} x \backslash y & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 4 \end{array}.$$

So, $$\mathbb{E}[XY] = 1 \times \frac{1}{15} + 2 \times \frac{2}{15} + 2 \times \frac{4}{15} + 4 \times \frac{8}{15} = \frac{1+4+8+32}{15}$$

$$= \frac{45}{15} = 3.$$

Method 2.2: We can also work directly with the formula

$$\mathbb{E}[XY] = \sum_x \sum_y xy\, p_{X,Y}(x,y) = \sum_{x=1}^{2} \sum_{y=1}^{2} (c\,x^2 y)(xy)$$

$$= c \sum_{x=1}^{2} \sum_{y=1}^{2} x^3 y^2 = c \times \sum_{x=1}^{2} x^3 \sum_{y=1}^{2} y^2$$

$$= c\,(1^3+2^3)(1^2+2^2) = c \times (1+8) \times (1+4) = c \times 9 \times 5$$

$c = \frac{1}{15}$

$$= \frac{9 \times 5}{15} = 3.$$