3 Classical Probability

Classical probability, which is based upon the ratio of the number of outcomes favorable to the occurrence of the event of interest to the total number of possible outcomes, provided most of the probability models used prior to the 20th century. It is the first type of probability problems studied by mathematicians, most notably, Frenchmen Fermat and Pascal whose 17th century correspondence with each other is usually considered to have started the systematic study of probabilities. [18, p 3] Classical probability remains of importance today and provides the most accessible introduction to the more general theory of probability.

Definition 3.1. Given a finite sample space Ω, the ***classical probability*** of an event A is

$$P(A) = \frac{|A|}{|\Omega|} \tag{1}$$

[6, Defn. 2.2.1 p 58]. In traditional language, a probability is a fraction in which the bottom represents the number of possible outcomes, while the number on top represents the number of outcomes in which the event of interest occurs.

- Assumptions: When the following are not true, do not calculate probability using (1).
  - Finite Ω: The number of possible outcomes is finite.
  - Equipossibility: The outcomes have equal probability of occurrence.
- The bases for identifying equipossibility were often
  - physical symmetry (e.g. a well-balanced dice, made of homogeneous material in a cubical shape) or
  - a balance of information or knowledge concerning the various possible outcomes.
- Equipossibility is meaningful only for finite sample space, and, in this case, the evaluation of probability is accomplished through the definition of classical probability.

- We will NOT use this definition beyond this section. We will soon introduce a formal definition in Section 5.

- In many problems, when the finite sample space does not contain equally likely outcomes, we can redefine the sample space to make the outcome equipossible.

Ex. 9,999 red balls
1 black ball

Grab one ball.

Wrong sample space:
Two "possible" outcomes R, K
These outcomes are not equiprobable. Therefore, can't apply $P(A) = \frac{|A|}{|\Omega|}$.

Example 3.2 (Slide). In drawing a card from a deck, there are 52 equally likely outcomes, 13 of which are diamonds. This leads to a probability of 13/52 or 1/4.

3.3. Basic properties of classical probability: From Definition 3.1, we can easily verified[4] the properties below.

$P(A) = \frac{|A|}{|\Omega|}$

- $P(A) \geq 0$
- $P(\Omega) = 1$
- $P(\emptyset) = 0$
- $P(A^c) = 1 - P(A)$

$P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{|\Omega| - |A|}{|\Omega|}$

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ which comes directly from
$$|A \cup B| = |A| + |B| - |A \cap B|.$$
- $A \perp B \Rightarrow P(A \cup B) = P(A) + P(B)$
- Suppose $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $P(\{\omega_i\}) = \frac{1}{n}$. Then $P(A) = \sum_{\omega \in A} P(\{\omega\})$.
    - The probability of an event is equal to the sum of the probabilities of its component outcomes because outcomes are mutually exclusive

[4]Because we will not rely on Definition 3.1 beyond this section, we will not worry about how to prove these properties. In Section 5, we will prove the same properties in a more general setting.

Example 3.4 (Slides)**.** When rolling two dice, there are 36 (equiprobable) possibilities.

$$P[\text{sum of the two dice} = 5] = 4/36.$$

Though one of the finest minds of his age, Leibniz was not immune to blunders: he thought it just as easy to throw 12 with a pair of dice as to throw 11. The truth is...

5+6, 6+5 $\quad P[\text{sum of the two dice} = 11] =$ 2/36

6+6 $\quad P[\text{sum of the two dice} = 12] =$ 1/36

Definition 3.5. In the world of gambling, probabilities are often expressed by ***odds***. To say that the odds are n:1 *against* the event A means that it is n times as likely that A does not occur than that it occurs. In other words, $P(A^c) = nP(A)$ which implies $P(A) = \frac{1}{n+1}$ and $P(A^c) = \frac{n}{n+1}$.

"Odds" here has nothing to do with even and odd numbers. The odds also mean what you will win, in addition to getting your stake back, should your guess prove to be right. If I bet \$1 on a horse at odds of 7:1, I get back \$7 in winnings plus my \$1 stake. The bookmaker will break even in the long run if the probability of that horse winning is 1/8 (not 1/7). Odds are "even" when they are 1:1 - win \$1 and get back your original \$1. The corresponding probability is 1/2.

3.6. It is important to remember that classical probability relies on the assumption that the outcomes are ***equally likely***.

Example 3.7. ***Mistake*** made by the famous French mathematician Jean Le Rond d'Alembert (18th century) who is an author of several works on probability:

"The number of heads that turns up in those two tosses can be 0, 1, or 2. Since there are three outcomes, the chances of each must be 1 in 3."

Correction: Four possible outcomes:

HH

HT

TH

TT

$P[\#H = 0] = \frac{1}{4}$

$P[\#H = 1] = \frac{2}{4} = \frac{1}{2}$

$P[\#H = 2] = \frac{1}{4}$

Ex Monty Hall

① At the beginning of the game, all three doors are equally likely to conceal a car.

Probability that your first chosen door will conceal a car $= \frac{1}{3}$.

② The remaining two doors at the end does not have the same probability of concealing the car.

$\Rightarrow$ can't use classical probability.

$$
\begin{array}{ccc}
 & \text{start} & \\
\swarrow^{1/3} & & \searrow^{2/3} \\
\text{choose the correct door} & & \text{choose the wrong door.} \\
\downarrow & & \downarrow \\
\text{at the end of the game, the car is behind your originally chosen door.} & & \text{at the end of the game, the car is behind the door that you did not choose originally.}
\end{array}
$$

$\frac{999,999}{1,000,000}$ vs. $\frac{1}{1,000,000}$