ECS 315: Quiz 3 Solution

Instructions

1. Separate into groups of no more than three persons.
2. Only one submission is needed for each group. Late submission will not be accepted.
3. ***Write down all the steps*** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
4. **Do not panic.**

Name	ID
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1. Arrivals of call request to a mobile base station are modeled by a Poisson process with a rate of ~~λ~~ 10 requests per hour. Let N be the number of call requests made between 9:30 and 10:00. What is the probability that N > 1?

Counting occurrences of a Poisson process within a time interval of length T gives a Poisson RV with parameter $\alpha = \lambda T$.

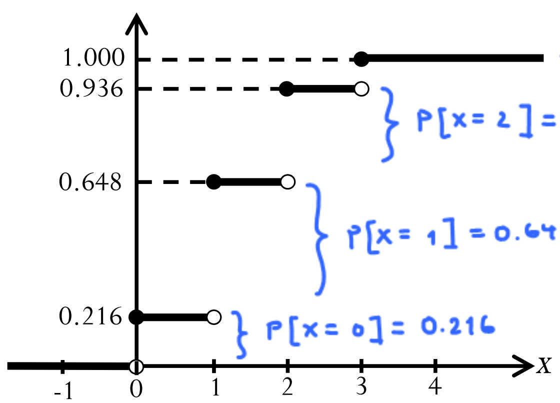
The duration of the time interval of interest is $T = \frac{1}{2}$ hr.

Here, $\lambda = 10$. So, $\alpha = \lambda T = 10 \times \frac{1}{2} = 5$.

In which case, $P[N=k] = e^{-\alpha}\frac{\alpha^k}{k!}$.

$$P[N>1] = 1 - P[N \le 1] = 1 - P[N=0] - P[N=1] = 1 - e^{-\alpha}(1+\alpha) = 1 - 6e^{-5} \approx 0.9596$$

2. Suppose the cdf of a random variable X is plotted below.



$P[X=3] = 1 - 0.936 = 0.064$

$P[X=2] = 0.936 - 0.648 = 0.288$

$P[X=1] = 0.648 - 0.216 = 0.432$

$P[X=0] = 0.216$

Find $\mathbb{E}X$.

The staircase-like cdf tells us that X is a discrete RV. The cdf plot also starts from 0 and ends at 1; so we know that all the "interesting" probabilities happen inside the given interval. Furthermore, we have notec before that we can get the pmf values from the locations and the sizes of the jumps in the cdf plot.

$$p_X(x) = \begin{cases} 0.216, & x = 0, \\ 0.432, & x = 1, \\ 0.288, & x = 2, \\ 0.064, & x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

$$\Rightarrow \mathbb{E}X = \sum_x x\,p_X(x) = 0 \times 0.216 + 1 \times 0.432 + 2 \times 0.288 + 3 \times 0.064 = 1.2$$

Quiz 4 Solution

Friday, September 20, 2013
10:33 AM

Suppose $p_{X,Y}(x,y) = \begin{cases} cx^2y, & x \in \{1,2\}, y \in \{1,2\}, \\ 0, & \text{otherwise.} \end{cases}$

① Find the joint pmf matrix $\mathbf{P}_{X,Y}$

First we identify the supports for X and Y.

Note that the given joint pmf is nonzero only when $x \in \{1,2\}$ and $y \in \{1,2\}$.

Therefore, we conclude that a support of X is $\{1,2\}$ and a support of Y is $\{1,2\}$.

We can then express the joint pmf using the joint pmf matrix:

$$\mathbf{P}_{X,Y} = \begin{array}{c|c} x \backslash y & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} c \times 1^2 \times 1 & c \times 1^2 \times 2 \\ c \times 2^2 \times 1 & c \times 2^2 \times 2 \end{bmatrix} \end{array} = \begin{array}{c|c} x \backslash y & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} c & 2c \\ 4c & 8c \end{bmatrix} \end{array}.$$

To find the constant c, recall that $\underbrace{\sum_x \sum_y}$ $p_{X,Y}(x,y) = 1$.

Here, we only need to focus on the (x,y)-pairs that are in the joint pmf matrix because other pairs have zero probability.

So, we need

$$c + 2c + 4c + 8c = 1$$
$$15c = 1$$
$$c = \frac{1}{15}.$$

Plugging this value in the joint pmf matrix above, we get

$$\mathbf{P}_{X,Y} = \begin{array}{c|c} x \backslash y & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 1/15 & 2/15 \\ 4/15 & 8/15 \end{bmatrix} \end{array}.$$

② Find $\mathbb{E}X$

To find $\mathbb{E}X$, we need $p_X(x)$. This can be obtained from the joint pmf matrix by $\underbrace{\text{summing along each row}}$.

This is the same as applying the formula $p_X(x) = \sum_y p_{X,Y}(x,y)$.

sum along each row

$$P_{X,Y} = \begin{array}{c|cc} x \backslash y & 1 & 2 \\ \hline 1 & 1/15 & 2/15 \\ 2 & 4/15 & 8/15 \end{array} \quad \begin{array}{l} \longrightarrow \; 3/15 = 1/5 \\ \longrightarrow \; 12/15 = 4/5 \end{array}$$

Therefore, we have $p_X(x) = \begin{cases} 1/5, & x = 1, \\ 4/5, & x = 2, \\ 0, & \text{otherwise}. \end{cases}$

This gives $\mathbb{E}X = \sum_x x\, p_X(x) = 1 \times \frac{1}{5} + 2 \times \frac{4}{5} = \frac{9}{5}$.

③ Are X and Y independent?

To check whether $X \perp\!\!\!\perp Y$, we need to show that

$$p_{X,Y}(x,y) = p_X(x)\, p_Y(y) \quad \underbrace{\text{for any } x, y.}$$

This can be reduced to only the x in a support of X and y in a support of Y because we automatically get $0 = 0 \times 0$ outside the supports.

We've already found $p_{X,Y}(x,y)$ and $p_X(x)$. So, we still need $p_Y(y)$. This can be obtained from summing along each column of the joint pmf matrix:

$$P_{X,Y} = \begin{array}{c|cc} x \backslash y & 1 & 2 \\ \hline 1 & 1/15 & 2/15 \\ 2 & 4/15 & 8/15 \\ & \downarrow & \downarrow \\ & 5/15 & 10/15 \\ & \| & \| \\ & 1/3 & 2/3 \end{array} \quad \Leftarrow \text{Sum along each column.}$$

So, we have $p_Y(y) = \begin{cases} 1/3, & y = 1, \\ 2/3, & y = 2, \\ 0, & \text{otherwise}. \end{cases}$

Now, to check for independence, we need to check whether

$$p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$$

for $x \in \{1,2\}$ and $y \in \{1,2\}$.

A concise way to do this is to express $p_X(x)$ and $p_Y(y)$

as row vectors $\underline{p}_X$ and $\underline{p}_Y$, respectively. Then check whether

$$(\underline{p}_X)^T \underline{p}_Y = \underline{P}_{X,Y}.$$

Here, $\underline{p}_X = [1/5 \quad 4/5]$ and $\underline{p}_Y = [\frac{1}{3} \quad \frac{2}{3}]$.

Therefore, $(\underline{p}_X)^T \underline{p}_Y = \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix} [\frac{1}{3} \quad \frac{2}{3}] = \begin{bmatrix} 1/15 & 2/15 \\ 4/15 & 8/15 \end{bmatrix}$.

This is exactly the same as the joint pmf matrix $\underline{P}_{X,Y}$ that we found in part 1. Therefore, $X \perp\!\!\!\perp Y$.

4 Find $\mathbb{E}[XY]$

Method 1: When $X \perp\!\!\!\perp Y$, we know that

$$\mathbb{E}[h(X)g(Y)] = \mathbb{E}[h(X)]\,\mathbb{E}[g(Y)],$$

for "any" functions h and g.

In particular, $\mathbb{E}[XY] = \underbrace{(\mathbb{E}X)}_{\frac{9}{5}}\underbrace{(\mathbb{E}Y)}_{\frac{1}{3}\times 1 + \frac{2}{3}\times 2 = \frac{5}{3}} = \frac{9}{5}\times\frac{5}{3} = 3$.

Method 2: We can also use our general formula (LOTUS):

$$\mathbb{E}[g(X,Y)] = \sum_x \sum_y g(x,y)\, p_{X,Y}(x,y).$$

This formula works regardless of whether $X \perp\!\!\!\perp Y$.

Here,

$$\mathbb{E}[XY] = \sum_x \sum_y xy\, p_{X,Y}(x,y).$$

Method 2.1:

For conciseness, we first find a matrix that contains the values of xy:

$$\begin{array}{c|c} x \backslash y & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 1\times 1 & 1\times 2 \\ 2\times 1 & 2\times 2 \end{bmatrix} \end{array} = \begin{array}{c|c} x \backslash y & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \end{array}.$$

So, $\mathbb{E}[XY] = 1\times\frac{1}{15} + 2\times\frac{2}{15} + 2\times\frac{4}{15} + 4\times\frac{8}{15} = \frac{1+4+8+32}{15}$

$= \frac{45}{15} = 3.$

Method 2.2: We can also work directly with the formula

$$\mathbb{E}[XY] = \sum_x \sum_y xy\, p_{X,Y}(x,y) = \sum_{x=1}^{2} \sum_{y=1}^{2} (c\,x^2 y)(xy)$$

$$= c \sum_{x=1}^{2} \sum_{y=1}^{2} x^3 y^2 = c \times \sum_{x=1}^{2} x^3 \sum_{y=1}^{2} y^2$$

$$= c\,(1^3+2^3)(1^2+2^2) = c \times (1+8) \times (1+4) = c \times 9 \times 5$$

$c = \frac{1}{15}$

$$= \frac{9 \times 5}{15} = 3.$$