13 Transform methods: Characteristic Functions

u, t

Definition 13.1. The characteristic function of a random variable X is defined by

$$\varphi_X(v) = \mathbb{E}\left[e^{jvX}\right].$$

if X is discrete $= \sum_x g(x)\, p_X(x) = \sum_x e^{jvx} p_X(x)$

$\mathbb{E}[g(X)]$

Remarks:

(a) If X is a continuous random variable with density f_X, then

$e^{jv0} \times (1-p) + e^{jv1} \times p$

$$\varphi_X(v) = \int_{-\infty}^{+\infty} e^{jvx} f_X(x)dx,$$

$X \sim \text{Bernoulli}(p)$

$\varphi_X(v) = 1-p+pe^{jv}$

which is the *Fourier transform* of f_X evaluated at $-v$. More precisely,

$$\varphi_X(v) = \mathcal{F}\left\{f_X\right\}(\omega)\big|_{\omega=-v}. \tag{38}$$

(b) Many references use u or t instead of v.

Example 13.2. You may have learned that the Fourier transform of a Gaussian waveform is a Gaussian waveform. In fact, when $X \sim \mathcal{N}(m, \sigma^2)$,

$$\mathcal{F}\left\{f_X\right\}(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{-j\omega x} dx = e^{-j\omega m - \frac{1}{2}\omega^2\sigma^2}.$$

Using (38), we have

$$\varphi_X(v) = e^{jvm - \frac{1}{2}v^2\sigma^2}.$$

Example 13.3. For $X \sim \mathcal{E}(\lambda)$, we have $\varphi_X(v) = \frac{\lambda}{\lambda - jv}$.

$$\varphi_X(v) = \mathbb{E}\left[e^{jvX}\right] = \int_{-\infty}^{\infty} e^{jvx} f_X(x)\, dx$$

$$= \int_0^{\infty} e^{jvx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-(\lambda - jv)x} dx$$

$$= \frac{\lambda}{\lambda - jv} \qquad \int_0^{\infty} e^{-\alpha x} dx = \frac{1}{\alpha}$$

As with the Fourier transform, we can build a large list of commonly used characteristic functions. (You probably remember that rectangular function in time domain gives a sinc function in frequency domain.) When you see a random variable that has the same form of characteristic function as the one that you know, you can quickly make a conclusion about the family and parameters of that random variable.

Example 13.4. Suppose a random variable X has the characteristic function $\varphi_X(v) = \frac{2}{2-jv}$. You can quickly conclude that it is an exponential random variable with parameter 2.

For many random variables, it is easy to find its expected value or any moments via the characteristic function. This can be done via the following result.

13.5. $\varphi_X^{(k)}(v) = j^k \mathbb{E}\left[X^k e^{jvX}\right]$ and $\varphi_X^{(k)}(0) = j^k \mathbb{E}\left[X^k\right]$.

$\varphi_X(v) = \mathbb{E}\left[e^{jvX}\right]$

$\frac{d}{dv}\varphi_X(v) = \mathbb{E}\left[jX e^{jvX}\right]$

$\frac{d^2}{dv^2}\varphi_X(v) = \mathbb{E}\left[(jX)^2 e^{jvX}\right]$

$\varphi_X''(0) = -\mathbb{E}\left[X^2\right]$

$\varphi_X'(0) = j\mathbb{E}X$

Example 13.6. When $X \sim \mathcal{E}(\lambda)$,

$\varphi_X(v) = \frac{\lambda}{\lambda - jv}$

(a) $\mathbb{E}X = \frac{1}{\lambda}$.

$\varphi_X'(v) = -\frac{\lambda}{(\lambda - jv)^2}(-j)$

$\varphi_X(0) = j \times \frac{1}{\lambda} = j\mathbb{E}X$

(b) $\operatorname{Var} X = \frac{1}{\lambda^2}$.

$\varphi_X''(v) = \cancel{-}\lambda(\cancel{-}j)(\cancel{-}2)(\lambda - jv)^{-3}(\cancel{-}j)$

$\operatorname{Var} X = \left(\frac{2}{\lambda^2}\right) - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$

$\varphi_X''(0) = (-1) \times \frac{1}{\lambda^2} \times 2 = \frac{-2}{\lambda^2} \Rightarrow \mathbb{E}\left[X^2\right] = \frac{2}{\lambda^2}$

$\sigma_X = \frac{1}{\lambda} = \mathbb{E}X$

Exercise 13.7 (F2011)**.** Continue from Example 13.2.

(a) Show that for $X \sim \mathcal{N}(m, \sigma^2)$, we have

(i) $\mathbb{E}X = m$

(ii) $\mathbb{E}\left[X^2\right] = \sigma^2 + m^2$.

(b) for $X \sim \mathcal{N}(3, 4)$, find $\mathbb{E}\left[X^3\right]$.

One very important properties of characteristic function is that it is very easy to find the characteristic function of a sum of independent random variables.

13.8. Suppose $X \perp\!\!\!\perp Y$. Let $Z = X + Y$. Then, the characteristic function of Z is the product of the characteristic functions of X and Y:

$$\varphi_Z(v) = \varphi_X(v)\varphi_Y(v)$$

$$\varphi_Z(v) = \mathbb{E}\left[e^{jvZ}\right] \overset{Z=X+Y}{=} \mathbb{E}\left[e^{jv(X+Y)}\right] = \mathbb{E}\left[\underbrace{e^{jvX}}_{g(X)} \underbrace{e^{jvY}}_{h(Y)}\right] \overset{X \perp\!\!\!\perp Y}{=} \mathbb{E}\left[e^{jvX}\right]\mathbb{E}\left[e^{jvY}\right]$$

Remark: Can you relate this property to the property of the Fourier transform?

Example 13.9. Use 13.8 to show that the sum of two independent Gaussian random variables is still a Gaussian random variable:

$X \sim \mathcal{N}(m_X, \sigma_X^2)$ $\quad X \perp\!\!\!\perp Y$ $\quad Z = X + Y$

$Y \sim \mathcal{N}(m_Y, \sigma_Y^2)$

$$\varphi_Z(v) = \varphi_X(v)\varphi_Y(v) = e^{jv(m_X+m_Y) - \frac{1}{2}v^2(\sigma_X^2+\sigma_Y^2)}$$

$\varphi_X(v) = e^{jvm_X - \frac{1}{2}v^2\sigma_X^2}$

$\varphi_Y(v) = e^{jvm_Y - \frac{1}{2}v^2\sigma_Y^2}$

$\Rightarrow Z \sim \mathcal{N}(m_X+m_Y, \sigma_X^2+\sigma_Y^2)$

Exercise 13.10. Continue from Example 11.44. Suppose $\Lambda_1 \sim \mathcal{P}(\lambda_1)$ and $\Lambda_2 \sim \mathcal{P}(\lambda_2)$ are independent. Let $\Lambda = \Lambda_1 + \Lambda_2$. Use 13.8 to show that $\Lambda \sim \mathcal{P}(\lambda_1 + \lambda_2)$.

Exercise 13.11. Continue from Example 11.45 Suppose $B_1 \sim \mathcal{B}(n_1, p)$ and $B_2 \sim \mathcal{B}(n_2, p)$ are independent. Let $B = B_1 + B_2$. Use 13.8 to show that $B \sim \mathcal{B}(n_1 + n_2, p)$.