ECS 315: Probability and Random Processes **2010/1**

EXAM 1 — Name ID

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Instructions

(a) Including this cover page, there are 12 pages. The last page is the formula sheet.

(b) Read these instructions and the questions carefully.

(c) Closed book. Closed notes.

(d) Basic calculators, e.g. FX-991MS, are permitted, but borrowing is not allowed.

(e) Allocate your time wisely. Some easy questions give many points.

(f) The use of communication devices including mobile phones is prohibited in the examination room. Put all of your communication devices in your bag and leave it at the front of the examination room.

(g) Do not forget to write your first name and the last three digits of your ID in the spaces provide on the top of each examination page, starting from page 2.

(h) Write down all the steps that you have done to obtain your answers. You may not get any credit even when your final answer is correct without showing how you get your answer.

(i) Some points are reserved for reducing answers into their simplest forms.

(j) Do not cheat.

(k) Do not panic.

Problem 1. (18 pt) In an experiment, A, B, C, and D are events with probabilities $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{8}$, $P(C) = \frac{5}{8}$, and $P(D) = \frac{3}{8}$. Furthermore, A and B are disjoint, while C and D are independent.

definition for disjoint sets

$A \cap B = \emptyset$

(a) Find

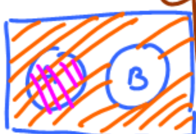
(i) (2 pt) $P(A \cap B) = P(\emptyset) = 0$

a property that we proved in class.

(ii) (2 pt) $P(A \cup B) =$
- $P(A) + P(B) - P(A \cap B)$
- $P(A) + P(B)$ when A, B are disjoint (finite additivity)

$= \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

(iii) (2 pt) $P(A \cap B^c) = P(A) = \frac{1}{4}$



A, B are disjoint

$A \cap B^c = A$

in class

$P(A \cap B^c) = P(A \setminus B)$

$= P(A) - P(A \cap B)$

$= \frac{1}{4} - 0 = \frac{1}{4}$

(iv) (2 pt) $P(A \cup B^c) = P(B^c) = 1 - P(B) = 1 - \frac{1}{8} = \frac{7}{8}$

A, B are disjoint

$A \cup B^c = B^c$

$P(A \cup \underbrace{B^c}_{E}) = P(A \cup E)$

$= P(A) + P(E) - P(A \cap E)$

$= P(A) + P(B^c) - P(A \cap B)$

$= P(B^c) = 1 - \frac{1}{8} = \frac{7}{8}$.

(b) (1 pt) Are A and B independent?

$P(A|B) \overset{?}{\neq} P(A)$

$0 = \frac{P(A \cap B)}{P(B)} \overset{?}{\neq} P(A) = \frac{1}{4}$

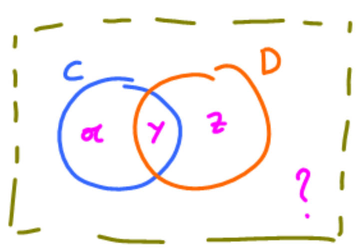
No, A and B are $\underbrace{\text{not independent}}_{\text{dependent}}$.

(c) (2 pt) Note that $P(C) + P(D) = 1$. Does this mean $D = C^c$? Justify your answer.

$P(C^c) = 1 - P(C)$

$P(C) + P(C^c) = 1$

If $D = C^c$, then $C \cap D = \emptyset$. No. $D \neq C^c$



$\alpha + y = \frac{5}{8}$

$y + z = \frac{3}{8}$

(d) Find

$P(C|D) = P(C)$

$\frac{P(C \cap D)}{P(D)} = P(C)$

$P(C \cap D) = P(C)P(D) = \frac{5}{8} \cdot \frac{3}{8} > 0$

(i) (2 pt) $P(C \cap D)$

(ii) (2 pt) $P(C \cap D^c)$

(iii) (2 pt) $P(C^c \cap D^c)$

(e) (1 pt) Are C^c and D^c independent?

Problem 2. (10 pt) Specify whether each of the following statements is TRUE or FALSE. If it is FALSE, provide your counter-example or explain why it is FALSE.

(a) For any events A, B, and C, if $A \perp B$ and $B \perp C$, then $A \perp C$. False

Counter-example

(b) If $\underbrace{P(A \cup B) = P(A) + P(B)}_{P(A \cap B) = 0}$, then $\overbrace{A \text{ and } B \text{ are disjoint}}^{A \cap B = \emptyset}$.

$P(E) = 0$ does not imply $E = \emptyset$

E could have element(s) with 0 probability.

(c) If $A \perp\!\!\!\perp B$, then $P(A) = P(A \cap B^c) + P(A)P(B)$.

(d) For any events A, B, and C, if $A \perp\!\!\!\perp B$ and $B \perp\!\!\!\perp C$, then $A \perp\!\!\!\perp C$.

(e) For any events A, B, and C, if $A \perp\!\!\!\perp B$, $B \perp\!\!\!\perp C$, and $A \perp\!\!\!\perp C$, then the events A, B, and C are independent.

Problem 3. (12 pt) Roll a fair six-sided dice five times. Let X_i be the number of dots that show up on the ith roll.

(a) (4 pt) List all $(X_1, X_2, X_3, X_4, X_5)$ where $X_i \in \{1, 2, 3, 4, 5, 6\}$ such that $X_1 + X_2 + X_3 + X_4 + X_5 = 6$. There should be 5 of these.

X_1	X_2	X_3	X_4	X_5
2	1	1	1	1
1	2	1	1	1
1	1	2	1	1
1	1	1	2	1
1	1	1	1	2

(b) (4 pt) What is the probability that $X_1 + X_2 + X_3 + X_4 + X_5 = 6$?

$$\frac{5}{6^5} = \frac{5}{7776} \simeq 6.43 \times 10^{-4}$$

(c) (2* pt) What is the probability that $X_1 + X_2 + X_3 + X_4 + X_5 = 10$?

1 1 1 1 1

$$\frac{\binom{9}{5}}{6^5}$$

(d) (2 pt) Given that $X_1 + X_2 + X_3 + X_4 + X_5 = 6$, find the probability that $X_1 = 1$.

Problem 4. (6 pt) Suppose that for the Country of Oz, 1 in 1000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 95% of the time. We would like to find the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive.

(a) (2 pt) What is $P(-|H)$, the conditional probability that a person tests negative given that the person does have the HIV virus?

(b) (2 pt) ***Use*** the law of total probability to find $P(+)$, the probability that a randomly chosen person tests positive. Provide at least 3 significant digits in your answer.

(c) (2 pt) ***Use*** Bayes' formula to find $P(H|+)$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive. Provide at least 3 significant digits in your answer.

Problem 5. (33 pt) The random variable V has pmf

$$p_V(v) = \begin{cases} \frac{1}{v^2} + c, & v \in \{-2, 2, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

(a) (5 pt) Find the value of the constant c.

$$\sum p_V = 1 \quad \Rightarrow \quad \left(\frac{1}{(-2)^2} + c + \frac{1}{2^2} + c + \frac{1}{3^2} + c \right) = 1$$

$$\cancel{3}c = \frac{1}{3}\left(1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{9}\right)$$

$$= \boxed{\frac{7}{54}} \approx 0.1296$$

(b) (2 pt) Find $P[V > 3]$. $= 0$

All the possible values of V are ≤ 3.

(c) (2 pt) Find $P[V < 3]$. $= P[V = -2] + P[V = 2]$

$$= \frac{1}{4} + c + \frac{1}{4} + c = \frac{1}{2} + \frac{14}{54} = \frac{41}{54}$$

(d) (2 pt) Find $P[V^2 > 1]$. $= 1$

v	v^2	
-2	4	>1
2	4	>1
3	9	>1

All possible values of V satisfy $(\cdot)^2 > 1$.

(e) (3 pt) Sketch $p_V(v)$. Provide as much information on the sketch as you can.

(f) (3 pt) Sketch $F_V(v)$. Provide as much information on the sketch as you can.

(g) (4 pt) Let $W = V^2 - V + 1$. Find the pmf of W.

v	$p_V(v)$	$v^2 - v + 1$
-2	$\frac{1}{4} + \frac{7}{54}$	7
2	$\frac{1}{4} + \frac{7}{54}$	3
3	$\frac{1}{9} + \frac{7}{54}$	7

$$p_W(w) = \begin{cases} \frac{41}{108}, & w = 3 \\ \frac{67}{108}, & w = 7 \\ 0, & \text{otherwise} \end{cases}$$

~~(h)~~ (3 pt) Find $\mathbb{E}V$

~~(i)~~ (3 pt) Find $\mathbb{E}[V^2]$

~~(j)~~ (3 pt) Find $\operatorname{Var} V$

(k) (1 pt) Find σ_V

(l) (2 pt) Find $\mathbb{E}W$

Problem 6. (16 pt) The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

$$\begin{array}{c c} x \backslash y & \begin{array}{ccc} 2 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{bmatrix} 0.02 & 0.10 & 0.08 \\ 0.08 & 0.32 & 0.40 \end{bmatrix} \end{array}$$

(a) (2 pt) Find the marginal pmf $p_X(x)$.

(b) (2 pt) Find the marginal pmf $p_Y(y)$.

(c) (2 pt) Find $\mathbb{E}X$

(d) (2 pt) Find $P[X = Y]$

(e) (2 pt) Find $P[XY < 6]$

(f) (2 pt) Find $\mathbb{E}\left[(X-3)(Y-2)\right]$

(g) (2 pt) Find $\mathbb{E}\left[X(Y^3 - 11Y^2 + 38Y)\right]$

(h) (2 pt) Are X and Y independent?

Problem 7. (2 pt) A random variables X has support containing only two numbers. Its expected value is $\mathbb{E}X = 5$. Its variance is $\operatorname{Var} X = 3$. Given an example of the pmf of such a random variable.

Problem 8. (2 pt) Suppose $X_1 \sim \text{Bernoulli}(1/3)$ and $X_2 \sim \text{Bernoulli}(1/4)$. Assume that $X_1 \perp\!\!\!\perp X_2$.

(a) (1 pt) Find the joint pmf matrix of the pair (X_1, X_2).

(b) (1 pt) Find the pmf of $Y = X_1 + X_2$.

Problem 9. (1 pt) Suppose X and Y are i.i.d. random variables. Suppose $\operatorname{Var} X = 5$ Find $\mathbb{E}\left[(X - Y)^2\right]$.