ECS 315: Probability and Random Processes **2012/1**

HW Solution 9 — Due: N/A

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Problem 1. A random variable X is a Gaussian random variable if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2},$$

for some constants m and σ. Furthermore, when a Gaussian random variable has $m = \sigma = 1$, we say that it is a standard Gaussian random variable. There is no closed-form expression for the cdf of the standard Gaussian random variable. The cdf itself is denoted by Φ and its values (or its complementary values $Q(\cdot) = 1 - \Phi(\cdot)$) are traditionally provided by a table. We refer to this kind of table as the Φ table. Examples of such tables are Table 3.1 and Table 3.2 in [Y&G].

Suppose Z is a standard Gaussian random variable.

(a) Use the Φ table to find the following probabilities:

(i) $P[Z < 1.52]$

(ii) $P[Z < -1.52]$

(iii) $P[Z > 1.52]$

(iv) $P[Z > -1.52]$

(v) $P[-1.36 < Z < 1.52]$

(b) Use the Φ table to find the value of c that satisfies each of the following relation.

(i) $P[Z > c] = 0.14$

(ii) $P[-c < Z < c] = 0.95$

Solution:

(a)

(i) $P[Z < 1.52] = \Phi(1.52) = \boxed{0.9357.}$

(ii) $P[Z < -1.52] = \Phi(-1.52) = 1 - \Phi(1.52) = 1 - 0.9357 = \boxed{0.0643.}$

(iii) $P[Z > 1.52] = 1 - P[Z < 1.52] = 1 - 0.9357 = \boxed{0.0643.}$

(iv) It is straightforward to see that the area of $P[Z > -1.52]$ is the same as $P[Z < 1.52] = \Phi(1.52)$. So, $P[Z > -1.52] = \boxed{0.9357.}$
Alternatively, $P[Z > -1.52] = 1 - P[Z \le -1.52] = 1 - \Phi(-1.52) = 1 - (1 - \Phi(1.52)) = \Phi(1.52)$.

(v) $P[-1.36 < Z < 1.52] = \Phi(1.52) - \Phi(-1.36) = \Phi(1.52) - (1 - \Phi(1.36)) = \Phi(1.52) + \Phi(1.36) - 1 = 0.9357 + 0.9131 - 1 = \boxed{0.8488.}$

(b)

(i) $P[Z > c] = 1 - P[Z \le c] = 1 - \Phi(c)$. So, we need $1 - \Phi(c) = 0.14$ or $\Phi(c) = 1 - 0.14 = 0.86$. In the Φ table, we do not have exactly 0.86, but we have 0.8599 and 0.8621. Because 0.86 is closer to 0.8599, we answer the value of c whose $\phi(c) = 0.8599$. Therefore, $c \approx \boxed{1.08.}$

(ii) $P[-c < Z < c] = \Phi(c) - \Phi(-c) = \Phi(c) - (1 - \Phi(c)) = 2\Phi(c) - 1$. So, we need $2\Phi(c) - 1 = 0.95$ or $\Phi(c) = 0.975$. From the Φ table, we have $c \approx \boxed{1.96.}$

Problem 2. The peak temperature T, as measured in degrees Fahrenheit, on a July day in New Jersey is a $\mathcal{N}(85, 100)$ random variable.

Remark: Do not forget that, for our class, the second parameter in $\mathcal{N}(\cdot, \cdot)$ is the variance (not the standard deviation).

(a) Express the cdf of T in terms of the Φ function

Hint: Recall that the cdf of a random variable T is given by $F_T(t) = P[T \le t]$. For $T \sim \mathcal{N}(m, \sigma^2)$,

$$F_T(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx.$$

The Φ function, which is the cdf of the standard Gaussian random variable, is given by

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

(b) Express each of the following probabilities in terms of the Φ function(s). Make sure that the arguments of the Φ functions are positive. (Positivity is required so that we can directly use the Φ/Q tables to evaluate the probabilities.)

(i) $P[T > 100]$

(ii) $P[T < 60]$

(iii) $P[70 \le T \le 100]$

(c) Express each of the probabilities in part (b) in terms of the Q function(s). Again, make sure that the arguments of the Q functions are positive.

(d) Evaluate each of the probabilities in part (b) using the Φ/Q tables.

(e) Observe that Table 3.1 stops at $z = 2.99$ and Table 3.2 starts at $z = 3.00$. Why is it better to give a table for $Q(z)$ instead of $\Phi(z)$ when z is large?

Solution:

(a) Continue from the hint. We perform a change of variable using $y = \frac{x-m}{\sigma}$ to get

$$F_T(t) = \int_{-\infty}^{\frac{x-m}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = \Phi\left(\frac{x-m}{\sigma}\right).$$

Here, $T \sim \mathcal{N}(85, 10^2)$. Therefore, $F_T(t) = \boxed{\Phi\left(\frac{t-85}{10}\right).}$

(b)

(i) $P[T > 100] = 1 - P[T \le 100] = 1 - F_T(100) = 1 - \Phi\left(\frac{100-85}{10}\right) = 1 - \Phi(1.5)$

(ii) $P[T < 60] = P[T \le 60]$ because T is a continuous random variable and hence $P[T = 60] = 0$. Now, $P[T \le 60] = F_T(60) = \Phi\left(\frac{60-85}{10}\right) = \Phi(-2.5) = \boxed{1 - \Phi(2.5).}$ Note that, for the last equality, we use the fact that $\Phi(-x) = 1 - \Phi(x)$.

(iii)

$$\begin{aligned} P[70 \le T \le 100] &= F_T(100) - F_T(70) = \Phi\left(\frac{100-85}{10}\right) - \Phi\left(\frac{70-85}{10}\right) \\ &= \Phi(1.5) - \Phi(-1.5) = \Phi(1.5) - (1 - \Phi(1.5)) = \boxed{2\Phi(1.5) - 1.} \end{aligned}$$

(c) In this question, we use the fact that $Q(x) = 1 - \Phi(x)$.

(i) $1 - \Phi(1.5) = \boxed{Q(1.5).}$

(ii) $1 - \Phi(2.5) = \boxed{Q(2.5).}$

(iii) $2\Phi(1.5) - 1 = 2(1 - Q(1.5)) - 1 = 2 - 2Q(1.5) - 1 = \boxed{1 - 2Q(1.5).}$

(d)

(i) $1 - \Phi(1.5) = 1 - 0.9332 = \boxed{0.0668.}$

(ii) $1 - \Phi(2.5) = 1 - 0.99379 = \boxed{0.0062.}$

(iii) $2\Phi(1.5) - 1 = 2(0.9332) - 1 = \boxed{0.8664.}$

(e) When z is large, $\Phi(z)$ will start with 0.999... The first few significant digits will all be the same and hence not quite useful to be there.

Problem 3. (Function of Continuous Random Variable) Let $X \sim \mathcal{E}(5)$ and $Y = 2/X$. Find

(a) $F_Y(y)$.

(b) $f_Y(y)$.

(c) $\mathbb{E}Y$

Hint: Because $\frac{d}{dy}e^{-\frac{10}{y}} = \frac{10}{y^2}e^{-\frac{10}{y}} > 0$ for $y \neq 0$. We know that $e^{-\frac{10}{y}}$ is an increasing function on our range of integration. In particular, consider $y > 10/\ln(2)$. Then, $e^{-\frac{10}{y}} > \frac{1}{2}$. Hence,

$$\int_0^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}\frac{1}{2}dy = \int_{10/\ln 2}^\infty \frac{5}{y}dy.$$

Solution:

(a) Because $X > 0$, we know that $Y > 0$ and hence, $F_Y(y) = 0$ for $y \le 0$. For $y > 0$,

$$\begin{aligned} F_Y(y) &= P[Y \le y] = P\left[\frac{2}{X} \le y\right] = P\left[X \ge \frac{2}{y}\right] = 1 - F_X\left(\frac{2}{y}\right) \\ &= e^{-5\left(\frac{2}{y}\right)} = e^{\frac{-10}{y}} \end{aligned}$$

Hence,

$$F_Y(y) = \boxed{\begin{cases} e^{-\frac{10}{y}}, & y > 0 \\ 0, & y \le 0 \end{cases}}$$

(b) Because we have already derive the cdf in the previous part, we can find the pdf via the cdf by $f_Y(y) = \frac{d}{dy}F_Y(y)$. This gives f_Y at all points except at $y = 0$ which we will set f_Y to be 0 there. Hence,

$$f_Y(y) = \boxed{\begin{cases} \frac{10}{y^2}e^{-\frac{10}{y}}, & y > 0 \\ 0, & y \le 0. \end{cases}}$$

Remark: In general, for $Y = \frac{a}{X}$, $f_Y(y) = \left|\frac{a}{y^2}\right| f_X\left(\frac{a}{y}\right)$.

(c) From the hint, we have

$$\begin{aligned}\mathbb{E}Y &= \int_0^\infty y\frac{10}{y^2}e^{-\frac{10}{y}}\,dy = \int_0^\infty \frac{10}{y}e^{-\frac{10}{y}}\,dy \\ &> \int_{10/\ln 2}^\infty \frac{10}{y}e^{-\frac{10}{y}}\,dy > \int_{10/\ln 2}^\infty \frac{10}{y}\frac{1}{2}dy = \int_{10/\ln 2}^\infty \frac{5}{y}dy \\ &= 5\ln y\Big|_{10/\ln 2}^\infty = \infty.\end{aligned}$$

Therefore, $\mathbb{E}Y = \boxed{\infty}$.

Problem 4. Solve Q3.5.6 using Table 3.1 and/or Table 3.2 from [Yates & Goodman, 2005]: A professor pays 25 cents for each blackboard error made in lecture to the student who points out the error. In a career of n years filled with blackboard errors, the total amount in dollars paid can be approximated by a Gaussian random variable Y_n with expected value $40n$ and variance $100n$.

(a) What is the probability that Y_{20} exceeds 1000?

(b) How many years n must the professor teach in order that $P[Y_n > 1000] > 0.99$?

Solution: See the handwritten solution.

Problem 5 (Joint pdf)**.** Solve Q4.4.2a, Q4.4.2b, Q4.4.3a from [Yates & Goodman, 2005]

Hint: To find c, recall that joint pdf must integrate to 1. To find any probability specified by a condition involving two random variables, you need to integrate the joint pdf over the corresponding region (which is the region that contains all the points satisfying the condition).

Problem 6. Consider the function

$$g(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

The function g operates like a **half-wave rectifier** in that if a positive voltage x is applied, the output is $y = x$, while if a negative voltage x is applied, the output is $y = 0$. Suppose $Y = g(X)$, where $X \sim \mathcal{U}(-1, 1)$. Plot the cdf of Y.

Solution: See the handwritten solution. [Gubner, 2006, p. 197–198]

Problem 7 (Joint pdf to marginal pdf + Expectation)**.** Solve Q4.5.1 from [Yates & Goodman, 2005]:

Random variables X and Y have the joint pdf

$$f_{X,Y}(x, y) = \begin{cases} 1/2, & -1 \leq x \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the region of nonzero probability. (The region of nonzero probability is defined to be the region where the joint pdf is positive.)

(b) What is $P[X > 0]$?

(c) What is $f_X(x)$?

(d) What is $\mathbb{E}X$?

Solution: See the handwritten solution.

Problem 8 (Function of two continuous random variables)**.** Solve Q4.6.8, Q4.7.8, Q4.7.12 from [Yates & Goodman, 2005]

Solution: See the handwritten solution.

Problem 9 (Independence)**.** Solve Q4.11.1 from [Yates & Goodman, 2005]

Solution: See the handwritten solution.

Problem 10. A student has passed a final exam by supplying correct answers for 26 out of 50 multiple-choice questions. For each question, there was a choice of three possible answers, of which only one was correct. The student claims not to have learned anything in the course and not to have studied for the exam, and says that his correct answers are the product of guesswork. Use Table 3.1 and/or Table 3.2 from [Yates & Goodman, 2005] to determine whether you should believe him.

Solution: This problem can be approached as follows: take as hypothesis that the student did guess at all the answers and calculate the probability of identifying 26 or more correct answers through guesswork. If this probability is below a threshold value you have chosen in advance, you judge that the student is bluffing.

If all the answers are guessed at, then the number of correct answers can be seen as the number of successes in $n = 50$ independent trials of a Bernoulli experiment having a success probability of $p = 1/3$. The binomial probability model is thus applicable.

A generally useful method of determining whether 26 correct answers is exceptional is based on finding out how many standard deviations lie between the observed number of correct answers achieved and the expected number. To do so, a quick approach is to approximate the binomial distribution with parameters $n = 50$ and $p = 1/3$ by a normal distribution with expected value $np = 50/3 \approx 16.67$ and standard deviation $\sqrt{np(1-p)} = 10/3 \approx 3.33$. So, the observed value of 26 correct answers lies

$$\frac{26 - \frac{50}{3}}{\frac{10}{3}} = 2.8$$

standard deviations above the expected value.

Note that it is useful to remember as a rule of thumb that the probability of a normally distributed random variable taking on a value lying three or more standard deviations above the expected value is very small (the probability is 0.00135). To be more precise, from Table 3.1, we get $1 - \Phi(2.8) \approx 1 - 0.99744 = 0.0026$.

From the analysis above, we see that the probability of such a deviation occurring is quite small. There is very good reason, therefore, to suppose that the student is bluffing, and that he in fact did prepare for the exam.

Problem 11. Suppose X and Y are i.i.d. $\mathcal{E}(\lambda)$ random variables.

(a) Find the characteristic function of

  (i) $X + Y$

  (ii) $2X + 5Y$

(b) Are your answers in part (a) still belongs to the exponential families?

Solution:

(a) The characteristic function of X and Y is given by

$$\varphi_X(v) = \varphi_Y(v) = \frac{\lambda}{\lambda - jv}.$$

(i) Because $X \perp\!\!\!\perp Y$, we have $\varphi_{X+Y}(v) = \varphi_X(v)\varphi_Y(v) = \boxed{\left(\frac{\lambda}{\lambda - jv}\right)^2}$.

(ii) By definition,

$$\varphi_{2X+5Y}(v) = \mathbb{E}\left[e^{iv(2X+5Y)}\right].$$

Because $X \perp\!\!\!\perp Y$, we have

$$\begin{aligned}\varphi_{2X+5Y}(v) &= \mathbb{E}\left[e^{iv2X}\right]e^{iv5Y} = \mathbb{E}\left[e^{i(v2)X}\right]\mathbb{E}\left[e^{i(v5)Y}\right] = \varphi_X(2v)\,\varphi_X(5v)\\ &= \frac{\lambda}{\lambda - j2v}\frac{\lambda}{\lambda - j5v} = \boxed{\frac{\lambda^2}{\lambda^2 - 7jv\lambda - 10v^2.}}\end{aligned}$$

(b) Both of them does NOT belong to the exponential families. (To still be in the exponential family, the characteristic function needs to be of the form $\frac{c}{c-jv}$ for some constant c.)

Q4: Y&G 3.5.6

Wednesday, September 08, 2010
8:30 AM

Q 3.5.6

$$Y_n \sim \mathcal{N}(40n, 100n)$$

Remark: Notice that the expected value and the variance in the approximation above is proportional to n. This kind of approximation occurs naturally from the central limit theorem (CLT):
When we have a sum of i.i.d. RVs, the CLT suggests that we may approximate the sum by a Gaussian RV. If the individual RVs has expected value m and variance σ^2, we know that the sum of n of them must have expected value nm and variance $n\sigma^2$.

Recall: $\mathbb{E}[X_1+X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$

$\mathrm{Var}[X_1+X_2] = \mathrm{Var}\,X_1 + \mathrm{Var}\,X_2$

when X_1 and X_2 are uncorrelated.

In class, we have seen that when $X \sim \mathcal{N}(m, \sigma^2)$,

$F_X(x) \equiv P[X \le x]$ can be calculated from $\Phi\left(\frac{x-m}{\sigma}\right)$

(a) $Y_{20} \sim \mathcal{N}(\underbrace{40 \times 20}, \underbrace{100 \times 20})$

Here, $m = 800$ and $\sigma = \sqrt{2000}$

$$P[Y_{20} > 1000] = 1 - F_{Y_{20}}(1000) = 1 - \Phi\left(\frac{1000-800}{\sqrt{2000}}\right) \approx 1 - \Phi(4.472)$$

$$\approx Q(4.47) \approx 3.91 \times 10^{-6}$$

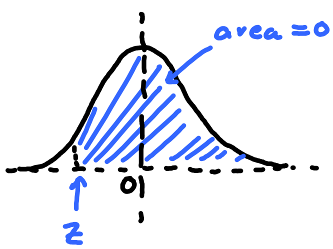
Table 3.2

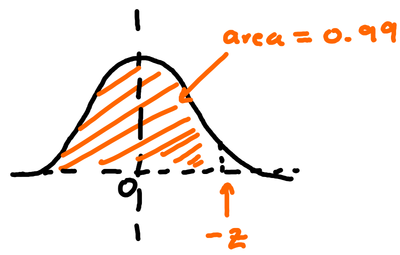
(b) We want to find n such that $P[Y_n > 1000] > 0.99$.

As in part (a), we start with $Y_n \sim \mathcal{N}(40n, 100n)$

$$P[Y_n > 1000] = 1 - F_{Y_n}(1000) = 1 - \Phi\Big(\underbrace{\frac{1000-40n}{10\sqrt{n}}}_{\equiv z}\Big)$$

First, we find z, such that $\underbrace{1-\Phi(z)}_{=\Phi(-z)} > 0.99$.





From Table 3.1, $\Phi(-z) > 0.99$ when $-z > 2.33$

(2.3263 if you have MATLAB)

Now, plugging in $z = \frac{1000-40n}{10\sqrt{n}}$, we have $\frac{4n-100}{\sqrt{n}} > 2.33$

Let $x = \sqrt{n}$. We then have

$$\frac{4x^2-100}{x} > 2.33$$

$$4x^2 - 2.33x - 100 > 0$$

$$\Rightarrow \cancel{x < -4.717} \text{ and } x > 5.3$$

negative

So, $n = x^2 > 28.1$ years

4.4.2

(a) We need $\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = 1$

This is the shorthand for $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$.

$\mathbb{R} = (-\infty, \infty)$

$\mathbb{R}^2 = (-\infty, \infty) \times (-\infty, \infty)$

Cartesian product.

After we exclude the pairs (x,y) whose $f_{X,Y}(x,y) = 0$, we need

$$\underbrace{\int_0^1\int_0^1 c x y^2 \,dx\,dy}_{} = 1$$

$$= c\int_0^1 x\,dx \int_0^1 y^2\,dy = c\left(\left.\frac{x^2}{2}\right|_0^1\right)\left(\left.\frac{y^3}{3}\right|_0^1\right)$$

$$= c \times \frac{1}{2} \times \frac{1}{3} = \frac{c}{6}$$

which means $c = 6$.

(b) Recall that $P[g(X,Y)\in B] = \iint_{R=\{(x,y):\, g(x,y)\in B\}} f_{X,Y}(x,y)\,dx\,dy$.

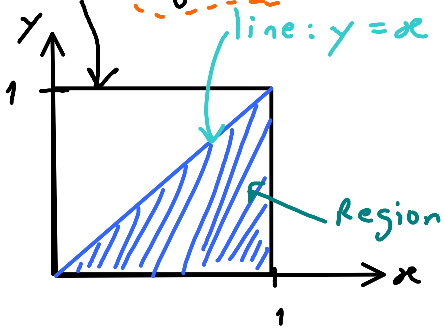
In particular, $P[X \geq Y] = \iint_{R=\{(x,y):\, x\geq y\}} f_{X,Y}(x,y)\,dx\,dy$.

Therefore,

to find $P[X\geq Y]$, we first need to find the "region" R where $x \geq y$. Because $f_{X,Y}(x,y)$ is nonzero only in $(x,y)\in[0,1]\times[0,1]$, we only need to focus on (x,y) in this square.

[We may call this square

$(x,y) \in [0,1]\times[0,1]$, we only need to focus on (x,y) in this square:



We may call this square "the region of nonzero density".
In Y&G, this is called "the region of nonzero probability."

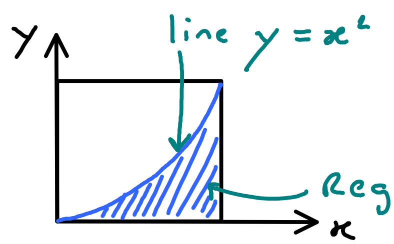
So, $$P[X \ge Y] = \iint_R f_{X,Y}(x,y)\,\underbrace{dy\,dx} = \int_0^1 \int_0^x c\,x y^2\,dy\,dx$$

I integrate w.r.t. y first because it is easier to write the limits of the integrations.

$$= c\int_0^1 x \int_0^x y^2\,dy\,dx = c\int_0^1 x \left.\frac{y^3}{3}\right|_0^x dx$$

$$= c\int_0^1 \frac{x^4}{3}\,dx = c\left.\frac{x^5}{15}\right|_0^1 = c\times\frac{1}{15} = \frac{6}{15} = \boxed{\frac{2}{5}}$$

The region R for $y < x^2$ is shown below



So, $$P[Y < X^2] = \int_0^1 \int_0^{x^2} c\,x y^2\,dy\,dx = c\int_0^1 x \int_0^{x^2} y^2\,dy\,dx$$

$$= c\int_0^1 x \left.\frac{y^3}{3}\right|_0^{x^2} dx = \frac{c}{3}\int_0^1 x^7\,dx = \frac{c}{3}\left.\frac{x^8}{8}\right|_0^1$$

$$= \frac{c}{24} = \boxed{\frac{1}{4}}$$

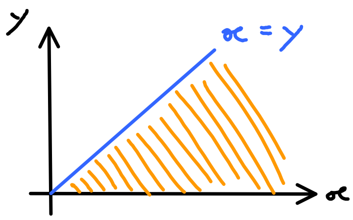
Q5: Q4.4.3

Wednesday, September 14, 2011
4:49 PM

4.4.3

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-2x}e^{-3y}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(Remark : Can you see right away that $X \sim \mathcal{E}(2)$, $Y \sim \mathcal{E}(3)$, and $X \perp\!\!\!\perp Y$.)



$$P[X > Y] = \int_0^\infty \int_0^x f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^\infty 2e^{-2x} \underbrace{\int_0^x 3e^{-3y}\,dy}_{\hookrightarrow P[Y < x] = 1 - P[Y \geq x]}$$

Recall that for $Y \sim \mathcal{E}(\lambda)$, $P[Y > y] = P[Y \geq y] = e^{-\lambda y}$

$$= \int_0^\infty 2e^{-2x}\left(1 - e^{-3x}\right)dx$$

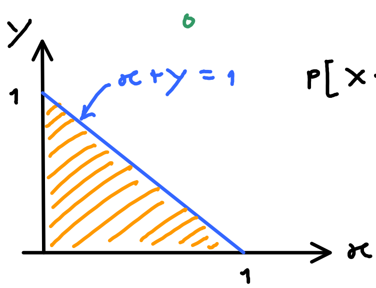
We can use the fact that pdf of exponential r.v. integrate to 1 to help us integrate other functions.

$$= \int_0^\infty 2e^{-2x}\,dx - \int_0^\infty 2e^{-5x}\,dx$$

$$= 1 - \frac{2}{5} = \boxed{\frac{3}{5}}$$

From $\int_0^\infty \lambda e^{-\lambda x}\,dx = 1$,

we have $\int_0^\infty e^{-\alpha x}\,dx = \frac{1}{\alpha}$



$$P[X + Y \leq 1] = \int_0^1 \int_0^{1-x} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^1 6e^{-2x} \int_0^{1-x} e^{-3y}\,dy\,dx$$

$$= \boxed{1 - 3e^{-2} + 2e^{-3}}$$

Q6

Friday, October 12, 2012
4:33 PM

As x varies over $[-1,1]$, $Y=g(X)$ varie over $[0,1]$.
This immediately gives us

$$F_Y(y) = \begin{cases} 0, & y<0 \\ 1, & y \geq 1. \end{cases}$$

Useful fact : If you know that your random variable always $\leq b$ and $\geq a$, then

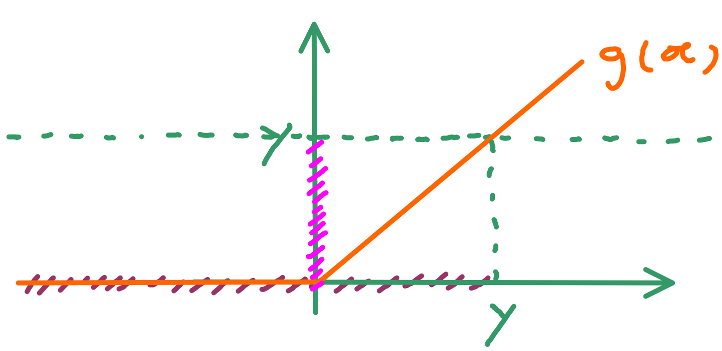
$$F_Y(y) = \begin{cases} 0, & y<a \\ 1, & y \geq b. \end{cases}$$

Of course, we still need to find $F_Y(y)$ for $0 \leq y < 1$.

To do this, consider the function g.

Observe that for $y \geq 0$,

$$g(x) \leq y \text{ if and only if } x \leq y.$$



Then,

$$F_Y(y) = P[Y \leq y] = P[X \leq y] = F_X(y)$$

$$= \underbrace{\frac{y-(-1)}{1-(-1)}} = \frac{y+1}{2}$$

formular for cdf of uniform r.v.

So,

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ (y+1)/2, & 0 \le y < 1 \\ 1, & y \geq 1. \end{cases}$$

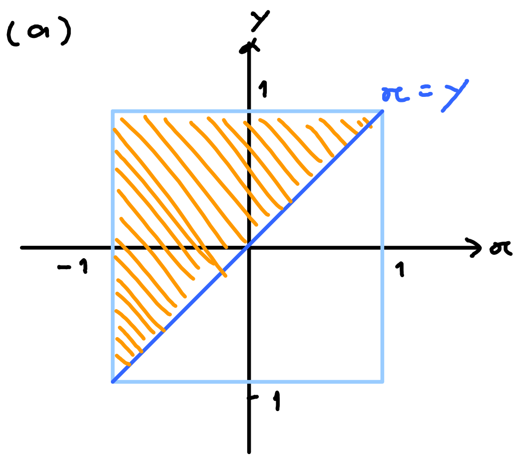
Q7: Y&G Q4.5.1

Wednesday, September 28, 2011
9:51 AM

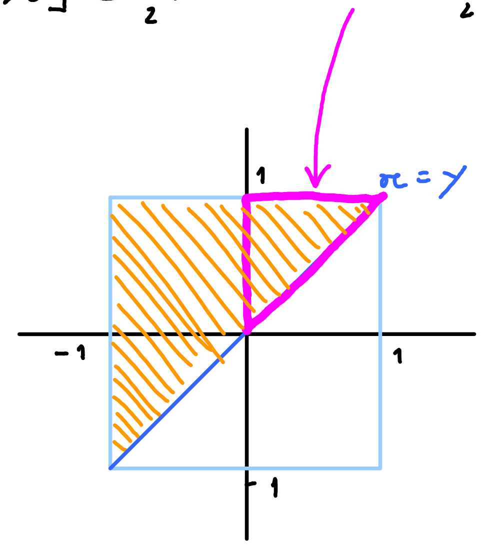
Q 4.5.1

$$f_{X,Y}(x,y) = \begin{cases} 1/2, & -1 \le x \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a)



(b) $P[X>0] = \frac{1}{2} \times \text{area here} = \frac{1}{2} \times \frac{1}{2} = \boxed{\frac{1}{4}}$



(c) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy$

First, we observe that $f_X(x) = 0$ for $x < -1$ or $x > +1$ because $f_{X,Y}(x,y) = 0$ for any y.

For $-1 \le x \le 1$,

$$f_X(x) = \int_{x}^{1} \frac{1}{2}\, dy = \frac{1}{2}(1-x)$$

Hence,

$$\boxed{f_X(x) = \begin{cases} \frac{1}{2}(1-x), & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}}$$

(d) $\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-1}^{1} x \frac{1}{2}(1-x)\, dx = \boxed{-\frac{1}{3}}$

Q8: Y&G Q4.6.8, Q4.7.8, Q4.7.12

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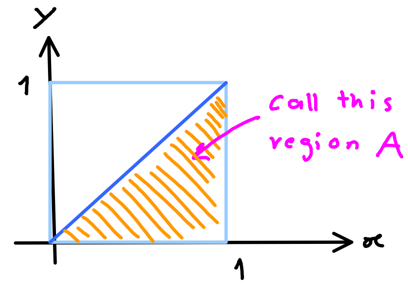
Q4.6.8

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

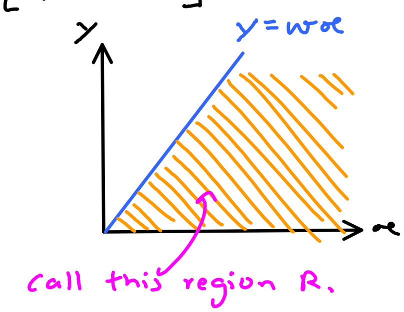


Note that both X and Y are nonnegative.

Hence, $W = \frac{Y}{X}$ is also nonnegative; that is $F_W(w) = 0$ for $w < 0$.

For $w > 0$, we have

$$F_W(w) = P[W \le w] = P\left[\frac{Y}{X} \le w\right] = P[Y \le wX]$$



Integrate $f_{X,Y}(x,y)$ over

Because (X,Y) is uniform over the region A,

$$F_W(w) = 2 \times (\text{area of } A \cap R)$$

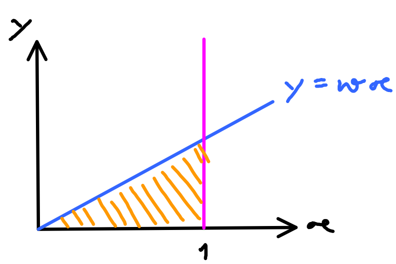
Note that when $w > 1$, $A \subset R$ which implies area of $A \cap R$

$= \text{area of } A$

$= \frac{1}{2}$

$$\text{So, } F_W(w) = 2 \times \frac{1}{2} = 1,$$

when $0 \le w \le 1$, $A \cap R$ is the region shown below with area $= \frac{1}{2} \times w \times 1 = \frac{w}{2}$.



$$\text{So, } F_W(w) = 2 \times \frac{w}{2} = w.$$

Hence,

$$\boxed{F_W(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \le w \le 1 \\ 1, & w > 1. \end{cases}}$$

From $f_W(w) = \frac{d}{dw} F_W(w)$, we have

$$\boxed{f_W(w) = \begin{cases} 1, & 0 \le w \le 1 \\ 0, & \text{otherwise.} \end{cases}}$$

So, W is uniform on $[0,1]$ and hence

$\boxed{S_W = [0,1]}$ and $\boxed{\mathbb{E}W = \frac{1}{2}}$

Q 4.7.8

(a) $f_X(x) = \begin{cases} \int_0^2 \frac{x+y}{3}\,dy, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{2}{3}(x+1), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

$\mathbb{E}X = \int_0^1 x \frac{2}{3}(x+1)\,dx = \boxed{\frac{5}{9}}$ $\quad \mathbb{E}X^2 = \int_0^1 x^2 \frac{2}{3}(x+1)\,dx = \frac{7}{18}$

$\operatorname{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \boxed{\frac{13}{162}} \approx 0.0802$

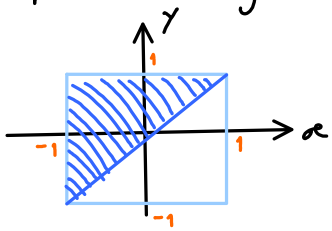
$$\boxed{\begin{array}{lll} \text{(a)} & \mathbb{E}X = \frac{5}{9}, & \operatorname{Var} X = 13/162 \\ \text{(b)} & \mathbb{E}Y = \frac{11}{9}, & \operatorname{Var} Y = 23/81 \\ \text{(c)} & \operatorname{Cov}[X,Y] = -1/81 & \\ \text{(d)} & \mathbb{E}[X+Y] = \frac{16}{9} & \\ \text{(e)} & \operatorname{Var}[X+Y] = \frac{55}{162} & \end{array}}$$

I'm too lazy to integrate these by hand.

Please see the MATLAB code.

Q 4.7.12

First, we plot the region of possible pairs (x,y):



$\mathbb{E}[XY] = \iint xy\, f_{X,Y}(x,y)\,dx\,dy = \int_{-1}^{1}\int_{x}^{1} xy \frac{1}{2}\,dy\,dx = \boxed{0}$

$\mathbb{E}[e^{X+Y}] = \int_{-1}^{1}\int_{x}^{1} e^{x+y}\frac{1}{2}\,dy\,dx = \boxed{\frac{e^{-2}}{4} + \frac{e^{2}}{4} - \frac{1}{2}}$

Q9: Y&G Q4.11.1

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(a) Fact: $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$ ← See the proof of this in the posted calculus review

Therefore, $\int_{-\infty}^{\infty} e^{-x^2/8} dx = \sqrt{8\pi} = 2\sqrt{2\pi}$ and $\int_{-\infty}^{\infty} e^{-y^2/18} dy = \sqrt{18\pi} = 3\sqrt{2\pi}$

Hence, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = c \times 2\sqrt{2\pi} \times 3\sqrt{2\pi} = 12\pi c$

This needs to be 1. Thus, $c = \boxed{\frac{1}{12\pi}}$.

Alternatively, note that $f_{X,Y}(x,y)$ is in the form of a bivariate Gaussian r.v. with $\rho = 0$ because there is no $x*y$ term.

So, $\sigma_X = \sqrt{4} = 2$

$\sigma_Y = \sqrt{9} = 3$

and $c = \frac{1}{2\pi\sigma_X\sigma_Y} = \frac{1}{2\pi \times 2 \times 3} = \frac{1}{12\pi}$

(b) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \frac{c}{3\sqrt{2\pi}} e^{-x^2/8}$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \frac{c}{2\sqrt{2\pi}} e^{-y^2/18}$

Hence, $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ which implies $\boxed{X \perp\!\!\!\perp Y}$