ECS 315: Probability and Random Processes **2012/1**

HW 9 — Due: N/A

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Problem 1. A random variable X is a Gaussian random variable if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2},$$

for some constants m and σ. Furthermore, when a Gaussian random variable has $m = \sigma = 1$, we say that it is a standard Gaussian random variable. There is no closed-form expression for the cdf of the standard Gaussian random variable. The cdf itself is denoted by Φ and its values (or its complementary values $Q(\cdot) = 1 - \Phi(\cdot)$) are traditionally provided by a table. We refer to this kind of table as the Φ table. Examples of such tables are Table 3.1 and Table 3.2 in [Y&G].

Suppose Z is a standard Gaussian random variable.

(a) Use the Φ table to find the following probabilities:

(i) $P[Z < 1.52]$

(ii) $P[Z < -1.52]$

(iii) $P[Z > 1.52]$

(iv) $P[Z > -1.52]$

(v) $P[-1.36 < Z < 1.52]$

(b) Use the Φ table to find the value of c that satisfies each of the following relation.

(i) $P[Z > c] = 0.14$

(ii) $P[-c < Z < c] = 0.95$

Problem 2. The peak temperature T, as measured in degrees Fahrenheit, on a July day in New Jersey is a $\mathcal{N}(85, 100)$ random variable.

Remark: Do not forget that, for our class, the second parameter in $\mathcal{N}(\cdot, \cdot)$ is the variance (not the standard deviation).

(a) Express the cdf of T in terms of the Φ function

Hint: Recall that the cdf of a random variable T is given by $F_T(t) = P[T \le t]$. For $T \sim \mathcal{N}(m, \sigma^2)$,

$$F_T(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx.$$

The Φ function, which is the cdf of the standard Gaussian random variable, is given by

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

(b) Express each of the following probabilities in terms of the Φ function(s). Make sure that the arguments of the Φ functions are positive. (Positivity is required so that we can directly use the Φ/Q tables to evaluate the probabilities.)

(i) $P[T > 100]$

(ii) $P[T < 60]$

(iii) $P[70 \le T \le 100]$

(c) Express each of the probabilities in part (b) in terms of the Q function(s). Again, make sure that the arguments of the Q functions are positive.

(d) Evaluate each of the probabilities in part (b) using the Φ/Q tables.

(e) Observe that Table 3.1 stops at $z = 2.99$ and Table 3.2 starts at $z = 3.00$. Why is it better to give a table for $Q(z)$ instead of $\Phi(z)$ when z is large?

Problem 3. (Function of Continuous Random Variable) Let $X \sim \mathcal{E}(5)$ and $Y = 2/X$. Find

(a) $F_Y(y)$.

(b) $f_Y(y)$.

(c) $\mathbb{E}Y$

Hint: Because $\frac{d}{dy} e^{-\frac{10}{y}} = \frac{10}{y^2} e^{-\frac{10}{y}} > 0$ for $y \neq 0$. We know that $e^{-\frac{10}{y}}$ is an increasing function on our range of integration. In particular, consider $y > 10/\ln(2)$. Then, $e^{-\frac{10}{y}} > \frac{1}{2}$. Hence,

$$\int_0^{\infty} \frac{10}{y} e^{-\frac{10}{y}} dy > \int_{10/\ln 2}^{\infty} \frac{10}{y} e^{-\frac{10}{y}} dy > \int_{10/\ln 2}^{\infty} \frac{10}{y} \frac{1}{2} dy = \int_{10/\ln 2}^{\infty} \frac{5}{y} dy.$$

Problem 4. Solve Q3.5.6 using Table 3.1 and/or Table 3.2 from [Yates & Goodman, 2005]:

A professor pays 25 cents for each blackboard error made in lecture to the student who points out the error. In a career of n years filled with blackboard errors, the total amount in dollars paid can be approximated by a Gaussian random variable Y_n with expected value $40n$ and variance $100n$.

(a) What is the probability that Y_{20} exceeds 1000?

(b) How many years n must the professor teach in order that $P[Y_n > 1000] > 0.99$?

Problem 5 (Joint pdf)**.** Solve Q4.4.2a, Q4.4.2b, Q4.4.3a from [Yates & Goodman, 2005]

Hint: To find c, recall that joint pdf must integrate to 1. To find any probability specified by a condition involving two random variables, you need to integrate the joint pdf over the corresponding region (which is the region that contains all the points satisfying the condition).

Problem 6. Consider the function

$$g(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

The function g operates like a **half-wave rectifier** in that if a positive voltage x is applied, the output is $y = x$, while if a negative voltage x is applied, the output is $y = 0$. Suppose $Y = g(X)$, where $X \sim \mathcal{U}(-1, 1)$. Plot the cdf of Y.

Problem 7 (Joint pdf to marginal pdf + Expectation)**.** Solve Q4.5.1 from [Yates & Goodman, 2005]:

Random variables X and Y have the joint pdf

$$f_{X,Y}(x, y) = \begin{cases} 1/2, & -1 \leq x \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the region of nonzero probability. (The region of nonzero probability is defined to be the region where the joint pdf is positive.)

(b) What is $P[X > 0]$?

(c) What is $f_X(x)$?

(d) What is $\mathbb{E}X$?

Problem 8 (Function of two continuous random variables)**.** Solve Q4.6.8, Q4.7.8, Q4.7.12 from [Yates & Goodman, 2005]

Hint: $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$ for $\alpha > 0$.

Problem 9 (Independence)**.** Solve Q4.11.1 from [Yates & Goodman, 2005]

Problem 10. A student has passed a final exam by supplying correct answers for 26 out of 50 multiple-choice questions. For each question, there was a choice of three possible answers, of which only one was correct. The student claims not to have learned anything in the course and not to have studied for the exam, and says that his correct answers are the product of guesswork. Use Table 3.1 and/or Table 3.2 from [Yates & Goodman, 2005] to determine whether you should believe him.

Problem 11. Suppose X and Y are i.i.d. $\mathcal{E}(\lambda)$ random variables.

(a) Find the characteristic function of

  (i) $X + Y$

  (ii) $2X + 5Y$

(b) Are your answers in part (a) still belongs to the exponential families?