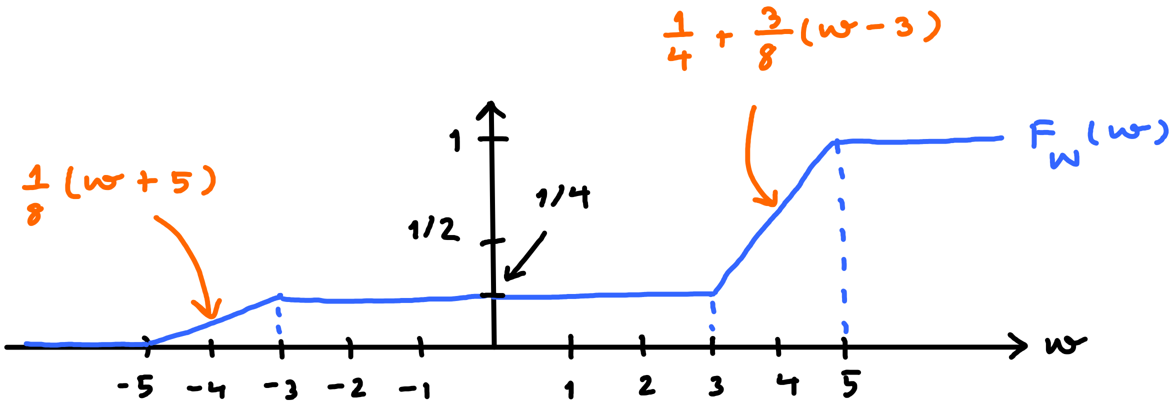
Y&G Q3.1.3

Wednesday, October 03, 2012
3:14 PM



(a) $P[W \le 4] = F_W(4) = \frac{1}{4} + \frac{3}{8}(4-3) = \frac{1}{4} + \frac{3}{8} = \frac{5}{8} \approx 0.625$

by definition of cdf

(b) $P[-2 < W \le 2] = F_W(2) - F_W(-2) = \frac{1}{4} - \frac{1}{4} = 0$

For continuous RV, $P[a \le X \le b] = F_X(b) - F_X(a)$

(c) $P[W > 0] = 1 - P[W \le 0] = 1 - F_W(0) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(A) = 1 - P(A^c)$

(d) $P[W \le a] = F_W(a)$. From the plot above, we know that to have $F_W(a) = \frac{1}{2}$, the value of a must be in the interval $(3,5)$. In this interval,

$$F_W(a) = \frac{1}{4} + \frac{3}{8}(a-3).$$

So, we solve for "a" that satisfies

$$\frac{1}{4} + \frac{3}{8}(a-3) = \frac{1}{2} \Rightarrow a = \frac{11}{3} \approx 3.67$$

Remark: It is possible to solve this problem by finding the pdf first. (I ask you to derive the pdf anyway in Q 3.2.3.)

However, you should also make sure that you know how to evaluate the probabilities above directly from the cdf.

Y&G Q3.2.1

Wednesday, October 03, 2012
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In this question, you are given a pdf whose expression has an unknown constant c.

(a) To find the constant c, recall that any pdf should integrate to 1.

In this problem,

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^2 cx\,dx = c\int_0^2 x\,dx$$

$$= c\left.\frac{x^2}{2}\right|_0^2 = \underbrace{2c}$$

This should $=1$.

Therefore, $c = \frac{1}{2}$.

(b) $P[0 \le X \le 1] = \int_0^1 f_X(x)\,dx = \int_0^1 \frac{1}{2}x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^1 = \frac{1}{4}$.

(c) $P\left[-\frac{1}{2} \le X \le \frac{1}{2}\right] = \int_{-1/2}^{1/2} f_X(x)\,dx = \int_0^{1/2} \frac{1}{2}x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^{1/2} = \frac{1}{16}$.

$f_X(x) = 0$ on $[-\frac{1}{2}, 0)$

(d) For $x<0$, because $f_X(t) = 0$ for $t<0$,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = 0$$

For $0 \le x \le 2$, $f_X(t) = \frac{t}{2}$ and

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \frac{t}{2}\,dt = \left.\frac{t^2}{4}\right|_0^x = \frac{x^2}{4}.$$

At $x=2$, $F_X(2) = 1$.

For $x > 2$, $f_X(t) = 0$. Therefore,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \underbrace{\int_{-\infty}^{2} f_X(t)\,dt}_{F_X(2)=1} + \underbrace{\int_{2}^{x} \underbrace{f_X(t)}_{0}\,dt}_{0} = 1.$$

Combining the three results above, we have

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x^2/4, & 0 \le x \le 2, \\ 1, & \text{otherwise}. \end{cases}$$

Y&G Q3.2.3

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Given a cdf, we can find the pdf by taking derivative.

As discussed in class, for the location(s) where derivative does not exist, we can choose to define the pdf to be any convenient value.

In this question, the cdf is given in the form of expressions on several intervals. It is then easy to find its derivative inside each of the intervals:

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} 0, & w < -5, \\ 1/8, & -5 < w < -3, \\ 0, & -3 < w < 3, \\ 3/8, & 3 < w < 5, \\ 0, & 5 < w. \end{cases}$$

It should be clear from the plot of cdf in Q3.1.3 that the derivative does not exist at $w = -5, -3, 3, 5$. We choose to assign $f_W(w) = 0$ at these points.

$$f_W(w) = \begin{cases} 1/8, & -5 < w < -3 \\ 3/8, & 3 < w < 5 \\ 0, & \text{otherwise} \end{cases}$$

Y&G Q3.3.4

Tuesday, September 07, 2010
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$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^2 \frac{y^2}{2} dy$$

For $y \notin [0,2)$,
$f_Y(y) = 0$ and hence
does not affect the integration

$$= \frac{1}{2} \frac{y^3}{3} \Big|_0^2 = \boxed{\frac{4}{3}}$$

$$\mathbb{E}[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^2 \frac{y^3}{2} dy$$

Again, we can ignore
the y whose $f_Y(y) = 0$.

$$= \frac{y^4}{2 \times 4} \Big|_0^2 = 2$$

$$\text{Var}\, Y = \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \boxed{\frac{2}{9}}$$

Y&G Q3.3.6

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We first find $f_V(v)$ by

$$f_V(v) = \frac{d}{dv} F_V(v) = \begin{cases} 0, & v < -5 \\ \frac{v+5}{72}, & -5 < v < 7 \\ 0, & v > 7 \end{cases}$$

Notice that I haven't specify $f_V(v)$ at $v = 5$ and $v = 7$. This is because the formula for $F_V(v)$ changes at those points and hence to actually find the derivatives, we would need to look at both the left and right derivatives at these points.

The good news is we don't have to find the derivative at $v = 5$ and $v = 7$. Recall that we can change some values of $f_V(v)$ without changing its property of being pdf. So, we can "define" or "set" $f_V(v)$ to be 0 at $v = 5$ and $v = 7$.

This gives

$$f_V(v) = \begin{cases} \frac{v+5}{72}, & -5 < v < 7 \\ 0, & \text{otherwise.} \end{cases}$$

(a)

$$\mathbb{E}V = \int_{-5}^{7} v \, \frac{v+5}{72} \, dv = \frac{1}{72} \int_{-5}^{7} v^2 + 5v \, dv = 3$$

(b)

$$^{2} \quad \int^{7} v^2 \, \frac{v+5}{\ } \, d$$

(b)

$$\mathbb{E}V^2 = \int_{-5}^{7} v^2 \frac{v+5}{72}\, dv = 17$$

$$\operatorname{Var} V = \mathbb{E}[V^2] - (\mathbb{E}V)^2 = \boxed{8}$$

(c)

$$\mathbb{E}V^3 = \int_{-5}^{7} v^3 \frac{v+5}{72}\, dv = \boxed{\frac{431}{5}} = 86.2$$

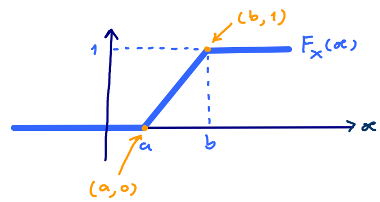
(a) $f_X(x) = \boxed{\begin{cases} \overset{\frac{1}{b-a}}{\frac{1}{10}}, & -5 < x < 5 \\ 0, & \text{otherwise} \end{cases}}$ $\quad a = -5,\ b = 5$

Alternatively, the above formula can be derived indirectly by thinking about how you would integrate $f_X(x)$ to get $F_X(x)$.

For $x < a$, there is no nonzero $f_X(x)$. Hence, $F_X(x) = 0$.

For $x > b$, we include all nonzero $f_X(x)$. Hence, $F_X(x) = 1$.

Finally, for $a \le x \le b$, we integrate a constant and hence we would get a straight line. It should start at $(a,0)$ and end at $(b,1)$ as shown below:



It should then be easy to find the equation governing the straight line passing through two points. You should get the same result as we got earlier.

(b) This might be a good place to try to find a general formula for $F_X(x)$ for $x \in (a,b)$.

For $x < a$,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0.$$

For $a \le x \le b$,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{a} f_X(t)\,dt + \int_{a}^{x} f_X(t)\,dt$$

$$= \int_{-\infty}^{a} 0\,dt + \int_{a}^{x} \frac{1}{b-a}\,dx = 0 + \frac{x-a}{b-a}$$

$$= \frac{x-a}{b-a}.$$

For $x > b$,

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{b} f(t)\,dt + \int_{b}^{x} f(t)\,dt$$

$$= F(b) + \int_{b}^{x} 0\,dt = \frac{b-a}{b-a} + 0 = 1.$$

Therefore,

$$\boxed{F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}}$$

(c) $$\mathbb{E}X = \int_a^b x \times \frac{1}{b-a}\,dx = \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

For $a = -5$ and $b = +5$,

$$\mathbb{E}X = \frac{5 + -5}{2} = \boxed{0}$$

(d) $$\mathbb{E}X^5 = \int_a^b x^5 \times \frac{1}{b-a}\,dx = \left.\frac{x^6}{6}\right|_a^b \times \frac{1}{b-a} = \frac{b^6 - a^6}{6(b-a)}$$

For $a = -5$ and $b = +5$,

$$\mathbb{E}X^5 = \boxed{0}$$

(e) $$\mathbb{E}e^X = \int_a^b e^x \times \frac{1}{b-a}\,dx = \frac{1}{b-a} \left. e^x \right|_a^b = \frac{e^b - e^a}{b-a}$$

For $a = -5$ and $b = +5$,

$$\mathbb{E}e^X = \boxed{\frac{e^5 - e^{-5}}{10}} \approx 14.84$$