

## HW Solution 7 — Due: Oct 4

*Lecturer: Prapun Suksompong, Ph.D.***Instructions**

- (a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.
- (b) It is important that you try to solve all problems. (5 pt)
- (c) Late submission will be heavily penalized.
- (d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
- (e) For problems that use **MATLAB**, you must also print and submit your m-file along with the numerical results. In the beginning part of your code, there should be one commented line that indicates your name and student ID.

**Problem 1.** The input  $X$  and output  $Y$  of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

	$y$	2	4	5
$x$				
1		0.02	0.10	0.08
3		0.08	0.32	0.40

- (a) Use **MATLAB** to evaluate the following quantities:
  - (i)  $\mathbb{E}[XY]$
  - (ii)  $\mathbb{E}[(X - 3)(Y - 2)]$
  - (iii)  $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)]$
  - (iv)  $\text{Cov}[X, Y]$
  - (v)  $\rho_{X,Y}$

- (b) Find  $\rho_{X,X}$
- (c) Calculate the following quantities using the values of  $\text{Var } X$ ,  $\text{Cov}[X, Y]$ , and  $\rho_{X,Y}$  that you got earlier.
- (i)  $\text{Cov}[3X + 4, 6Y - 7]$
  - (ii)  $\rho_{3X+4, 6Y-7}$
  - (iii)  $\text{Cov}[X, 6X - 7]$
  - (iv)  $\rho_{X, 6X-7}$

**Solution:**

- (a)
- (i) From MATLAB,  $\mathbb{E}[XY] = 11.16$
  - (ii) From MATLAB,  $\mathbb{E}[(X - 3)(Y - 2)] = -0.88$
  - (iii) From MATLAB,  $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)] = 104$
  - (iv) From MATLAB,  $\text{Cov}[X, Y] = 0.032$
  - (v) From MATLAB,  $\rho_{X,Y} = 0.044677$
- (b)  $\rho_{X,X} = \frac{\text{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\text{Var}[X]}{\sigma_X^2} = 1$
- (c)
- (i)  $\text{Cov}[3X + 4, 6Y - 7] = 3 \times 6 \times \text{Cov}[X, Y] \approx 3 \times 6 \times 0.032 \approx \boxed{0.576}$ .
  - (ii) Note that
 
$$\rho_{aX+b, cY+d} = \frac{\text{Cov}[aX + b, cY + d]}{\sigma_{aX+b} \sigma_{cY+d}}$$

$$= \frac{ac \text{Cov}[X, Y]}{|a| \sigma_X |c| \sigma_Y} = \frac{ac}{|ac|} \rho_{X,Y} = \text{sign}(ac) \times \rho_{X,Y}.$$
- Hence,  $\rho_{3X+4, 6Y-7} = \text{sign}(3 \times 6) \rho_{X,Y} = \rho_{X,Y} = \boxed{0.0447}$ .
- (iii)  $\text{Cov}[X, 6X - 7] = 1 \times 6 \times \text{Cov}[X, X] = 6 \times \text{Var}[X] \approx \boxed{3.84}$ .
  - (iv)  $\rho_{X, 6X-7} = \text{sign}(1 \times 6) \times \rho_{X,X} = \boxed{1}$ .

**Problem 2.** Suppose  $\text{Var } X = 5$ . Find  $\text{Cov}[X, X]$  and  $\rho_{X,X}$ .

**Solution:**

$$(a) \text{Cov}[X, X] = \mathbb{E}[(X - \mathbb{E}X)(X - \mathbb{E}X)] = \mathbb{E}[(X - \mathbb{E}X)^2] = \text{Var } X = \boxed{5}.$$

$$(b) \rho_{X,X} = \frac{\text{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\text{Var } X}{\sigma_X^2} = \frac{\text{Var } X}{\text{Var } X} = \boxed{1}.$$

**Problem 3.** The input  $X$  and output  $Y$  of a system subject to random perturbations are described probabilistically by the joint pmf  $p_{X,Y}(x, y)$ , where  $x = 1, 2, 3$  and  $y = 1, 2, 3, 4, 5$ . Let  $P$  denote the joint pmf matrix whose  $i, j$  entry is  $p_{X,Y}(i, j)$ , and suppose that

$$P = \frac{1}{71} \begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

- (a) Find  $\mathbb{E}[XY]$
- (b) Find  $\mathbb{E}[(3X + 4)(6Y - 7)]$
- (c) Find  $\text{Cov}[X, Y]$
- (d) Find  $\text{Cov}[3X + 4, 6Y - 7]$

**Solution:** All of the calculations in this question are simply plugging in numbers into appropriate formula.

- (a)  $\mathbb{E}[XY] = 6.0282$
- (b)  $\mathbb{E}[(3X + 4)(6Y - 7)] = 115.3239$
- (c)  $\text{Cov}[X, Y] = 0.0389$
- (d)  $\text{Cov}[3X + 4, 6Y - 7] = 3 \times 6 \times \text{Cov}[X, Y] = 0.6999$ .

**Problem 4.** Suppose we know that  $\sigma_X = \frac{\sqrt{21}}{10}$ ,  $\sigma_Y = \frac{4\sqrt{6}}{5}$ ,  $\rho_{X,Y} = -\frac{1}{\sqrt{126}}$ .

- (a) Find  $\text{Var}[X + Y]$ .
- (b) Find  $\mathbb{E}[(Y - 3X + 5)^2]$ . Assume  $\mathbb{E}[Y - 3X + 5] = 1$ .

**Solution:**

- (a) First, we know that  $\text{Var } X = \sigma_X^2 = \frac{21}{100}$ ,  $\text{Var } Y = \sigma_Y^2 = \frac{96}{25}$ , and  $\text{Cov}[X, Y] = \rho_{X,Y} \times \sigma_X \times \sigma_Y = -\frac{2}{25}$ . Now,

$$\begin{aligned} \text{Var}[X + Y] &= \mathbb{E} [((X + Y) - \mathbb{E}[X + Y])^2] = \mathbb{E} [((X - \mathbb{E}X) + (Y - \mathbb{E}Y))^2] \\ &= \mathbb{E} [(X - \mathbb{E}X)^2] + 2\mathbb{E} [(X - \mathbb{E}X)(Y - \mathbb{E}Y)] + \mathbb{E} [(Y - \mathbb{E}Y)^2] \\ &= \text{Var } X + 2\text{Cov}[X, Y] + \text{Var } Y \\ &= \boxed{\frac{389}{100}} = 3.89. \end{aligned}$$

Remark: It is useful to remember that

$$\text{Var}[X + Y] = \text{Var } X + 2\text{Cov}[X, Y] + \text{Var } Y.$$

Note that when  $X$  and  $Y$  are uncorrelated,  $\text{Var}[X + Y] = \text{Var } X + \text{Var } Y$ . This simpler formula also holds when  $X$  and  $Y$  are independence because independence is a stronger condition.

- (b) First, we write

$$Y - aX - b = (Y - \mathbb{E}Y) - a(X - \mathbb{E}X) - \underbrace{(a\mathbb{E}X + b - \mathbb{E}Y)}_c.$$

Now, using the expansion

$$(u + v + t)^2 = u^2 + v^2 + t^2 + 2uv + 2ut + 2vt,$$

we have

$$\begin{aligned} (Y - aX - b)^2 &= (Y - \mathbb{E}Y)^2 + a^2(X - \mathbb{E}X)^2 + c^2 \\ &\quad - 2a(X - \mathbb{E}X)(Y - \mathbb{E}Y) - 2c(Y - \mathbb{E}Y) + 2a(X - \mathbb{E}X)c. \end{aligned}$$

Recall that  $\mathbb{E}[X - \mathbb{E}X] = \mathbb{E}[Y - \mathbb{E}Y] = 0$ . Therefore,

$$\mathbb{E}[(Y - aX - b)^2] = \text{Var } Y + a^2 \text{Var } X + c^2 - 2a\text{Cov}[X, Y]$$

Plugging back the value of  $c$ , we have

$$\boxed{\mathbb{E}[(Y - aX - b)^2] = \text{Var } Y + a^2 \text{Var } X + (\mathbb{E}[(Y - aX - b)])^2 - 2a\text{Cov}[X, Y]}.$$

Here,  $a = 3$  and  $b = -5$ . Plugging these values along with the given quantities into the formula gives

$$\mathbb{E}[(Y - aX - b)^2] = \boxed{\frac{721}{100}} = 7.21.$$

**Problem 5.** Suppose  $X \sim \text{binomial}(5, 1/3)$ ,  $Y \sim \text{binomial}(7, 4/5)$ , and  $X \perp\!\!\!\perp Y$ . Use MATLAB to evaluate the following quantities.

(a)  $\mathbb{E}[(X - 3)(Y - 2)]$

(b)  $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)]$

(c)  $\text{Cov}[X, Y]$

(d)  $\rho_{X,Y}$

**Solution:**

(a)  $\mathbb{E}[(X - 3)(Y - 2)] = \boxed{-4.8}$

(b)  $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)] = \boxed{82.13}$

(c)  $\text{Cov}[X, Y] = \boxed{0}$  because  $X \perp\!\!\!\perp Y$ . (MATLAB should give a very small number.)

(d)  $\rho_{X,Y} = \boxed{0}$  because  $\text{Cov}[X, Y] = 0$  (MATLAB should give a very small number.)