

HW Solution 6 — Due: Sep 25

*Lecturer: Prapun Suksompong, Ph.D.***Instructions**

- (a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.
- (b) It is important that you try to solve all problems. (5 pt)
- (c) Late submission will be heavily penalized.
- (d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
- (e) For problems that use **MATLAB**, you must also print and submit your m-file along with the numerical results. In the beginning part of your code, there should be one commented line that indicates your name and student ID.

Problem 1. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

	y	2	4	5
x				
1		0.02	0.10	0.08
3		0.08	0.32	0.40

Use **MATLAB** or **Excel** to evaluate the following quantities:

- (a) The marginal pmf $p_X(x)$
- (b) The marginal pmf $p_Y(y)$
- (c) $\mathbb{E}X$
- (d) $\text{Var } X$

- (e) $\mathbb{E}Y$
- (f) $\text{Var } Y$
- (g) $P[XY < 6]$
- (h) $P[X = Y]$

Solution:

- (a) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 0.2, & x = 1 \\ 0.8, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

- (b) The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 0.1, & y = 2 \\ 0.42, & y = 4 \\ 0.48, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

- (c) From MATLAB or Excel, $\mathbb{E}X = 2.6$
- (d) From MATLAB or Excel, $\text{Var } X = 0.64$
- (e) From MATLAB or Excel, $\mathbb{E}Y = 4.28$
- (f) From MATLAB or Excel, $\text{Var } Y = 0.8016$
- (g) $P[XY < 6] = P[X = 1] = 0.2$
- (h) $P[X = Y] = 0$

Problem 2. The input X and output Y of a system subject to random perturbations are described probabilistically by the joint pmf $p_{X,Y}(x, y)$, where $x = 1, 2, 3$ and $y = 1, 2, 3, 4, 5$. Let P denote the joint pmf matrix whose i, j entry is $p_{X,Y}(i, j)$, and suppose that

$$P = \frac{1}{71} \begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

- (a) Find the marginal pmfs $p_X(x)$ and $p_Y(y)$.
- (b) Find $\mathbb{E}X$
- (c) Find $\mathbb{E}Y$
- (d) Find $\text{Var } X$
- (e) Find $\text{Var } Y$

Solution: All of the calculations in this question are simply plugging numbers into appropriate formula.

- (a) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 26/71, & x = 1 \\ 25/71, & x = 2 \\ 20/71, & x = 3 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.3662, & x = 1 \\ 0.3521, & x = 2 \\ 0.2817, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 13/71, & y = 1 \\ 8/71, & y = 2 \\ 21/71, & y = 3 \\ 15/71, & y = 4 \\ 14/71, & y = 5 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.1831, & y = 1 \\ 0.1127, & y = 2 \\ 0.2958, & y = 3 \\ 0.2113, & y = 4 \\ 0.1972, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

- (b) $\mathbb{E}X = \frac{136}{71} \approx 1.9155$
- (c) $\mathbb{E}Y = \frac{222}{71} \approx 3.1268$
- (d) $\text{Var } X = \frac{3230}{5041} \approx 0.6407$
- (e) $\text{Var } Y = \frac{9220}{5041} \approx 1.8290$

Problem 3. Suppose $X \sim \text{binomial}(5, 1/3)$, $Y \sim \text{binomial}(7, 4/5)$, and $X \perp\!\!\!\perp Y$.

- (a) A vector describing the pmf of X can be created by the MATLAB expression:

$$\mathbf{x} = 0:5; \text{ pX} = \text{binopdf}(\mathbf{x}, 5, 1/3).$$

What is the expression that would give pY , a corresponding vector describing the pmf of Y ?

- (b) Use `pX` and `pY` from part (a), how can you create the joint pmf matrix in MATLAB? Do not use “for-loop”, “while-loop”, “if statement”. Hint: Multiply them in an appropriate orientation.
- (c) Use **MATLAB** to evaluate the following quantities. Again, do not use “for-loop”, “while-loop”, “if statement”.
- $\mathbb{E}X$
 - $P[X = Y]$
 - $P[XY < 6]$

Solution: The MATLAB codes are provided in a separate file.

- (a) `y = 0:7; pY = binopdf(y,7,4/5);`
- (b) `P = pX.'*pY;`
- (c)
- $\mathbb{E}X = 1.667$
 - $P[X = Y] = 0.0121$
 - $P[XY < 6] = 0.2727$

Problem 4. A webpage server can handle r requests per day. Find the probability that the server gets more than r requests at least once in n days. Assume that the number of requests on day i is $X_i \sim \mathcal{P}(\alpha)$ and that X_1, \dots, X_n are independent.

Solution: [Gubner, 2006, Ex 2.10]

$$\begin{aligned}
 P\left[\bigcup_{i=1}^n [X_i > r]\right] &= 1 - P\left[\bigcap_{i=1}^n [X_i \leq r]\right] = 1 - \prod_{i=1}^n P[X_i \leq r] \\
 &= 1 - \prod_{i=1}^n \left(\sum_{k=0}^r \frac{\alpha^k e^{-\alpha}}{k!}\right) = 1 - \left(\sum_{k=0}^r \frac{\alpha^k e^{-\alpha}}{k!}\right)^n.
 \end{aligned}$$