ECS 315: Probability and Random Processes **2012/1**

HW Solution 4 — Due: Sep 6

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Instructions

(a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(b) It is important that you try to solve all problems. (5 pt)

(c) Late submission will be heavily penalized.

(d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Consider the sample space $\Omega = \{-2, -1, 0, 1, 2, 3, 4\}$. For an event $A \subset \Omega$, suppose that $P(A) = |A|/|\Omega|$. Define the random variable $X(\omega) = \omega^2$. Find the probability mass function of X.

Solution: Because $|\Omega| = 7$, we have $p(\omega) = 1/7$. The random variable maps the outcomes $-2, -1, 0, 1, 2, 3, 4$ to numbers $4, 1, 0, 1, 4, 9, 16$, respectively. Therefore,

$$\begin{aligned} p_X(0) &= P(\{0\}) = \frac{1}{7}, \\ p_X(1) &= P(\{-1, 1\}) = \frac{2}{7}, \\ p_X(4) &= P(\{-2, 2\}) = \frac{2}{7}, \\ p_X(9) &= P(\{3\}) = \frac{1}{7}, \text{ and} \\ p_X(16) &= P(\{4\}) = \frac{1}{7}. \end{aligned}$$

The pmf can then be expressed as

$$p_X(x) = \boxed{\begin{cases} \frac{1}{7}, & x = 0, 9, 16 \\ \frac{2}{7}, & x = 1, 4 \\ 0, & \text{otherwise.} \end{cases}}$$

Problem 2. Suppose X is a random variable whose pmf at $x = 0, 1, 2, 3, 4$ is given by $p_X(x) = \frac{2x+1}{25}$.

Remark: Note that the statement above does not specify the value of the $p_X(x)$ at the value of x that is not 0,1,2,3, or 4.

(a) What is $p_X(5)$?

(b) Determine the following probabilities:

(i) $P[X = 4]$

(ii) $P[X \le 1]$

(iii) $P[2 \le X < 4]$

(iv) $P[X > -10]$

Solution:

(a) First, we calculate

$$\sum_{x=0}^{4} p_X(x) = \sum_{x=0}^{4} \frac{2x+1}{25} = \frac{25}{25} = 1.$$

Therefore, there can't be any other x with $p_X(x) > 0$. At $x = 5$, we then conclude that $p_X(5) = \boxed{0.}$ The sam reasoning also implies that $p_X(x) = 0$ at any x that is not 0,1,2,3, or 4.

(b) Recall that, for discrete random variable X, the probability

$$P[\text{some condition(s) on } X]$$

can be calculated by adding $p_X(x)$ for all x in the support of X that satisfies the given condition(s).

(i) $P[X = 4] = p_X(4) = \frac{2\times4+1}{25} = \boxed{\frac{9}{25}.}$

(ii) $P[X \le 1] = p_X(0) + p_X(1) = \frac{2\times0+1}{25} + \frac{2\times1+1}{25} = \frac{1}{25} + \frac{3}{25} = \boxed{\frac{4}{25}.}$

(iii) $P[2 \le X < 4] = p_X(2) + p_X(3) = \frac{2\times2+1}{25} + \frac{2\times3+1}{25} = \frac{5}{25} + \frac{7}{25} = \boxed{\frac{12}{25}.}$

(iv) $P[X > -10] = \boxed{1}$ because all the x in the support of X satisfies $x > -10$.

Problem 3. The random variable V has pmf

$$p_V(v) = \begin{cases} cv^2, & v = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c.

(b) Find $P[V \in \{u^2 : u = 1, 2, 3, \ldots\}]$.

(c) Find the probability that V is an even number.

(d) Find $P[V > 2]$.

(e) Sketch $p_V(v)$.

(f) Sketch $F_V(v)$.

Solution: [Y&G, Q2.2.3]

(a) We choose c so that the pmf sums to one:

$$\sum_v p_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1.$$

Hence, $c = \boxed{1/30}$.

(b) $P[V \in \{u^2 : u = 1, 2, 3, \ldots\}] = p_V(1) + p_V(4) = c(1^2 + 4^2) = \boxed{17/30}$.

(c) $P[V \text{ even}] = p_V(2) + p_V(4) = c(2^2 + 4^2) = 20/30 = \boxed{2/3}$.

(d) $P[V > 2] = p_V(3) + p_V(4) = c(3^2 + 4^2) = 25/30 = \boxed{5/6}$.

(e) See Figure 4.1 for the sketch of $p_V(v)$:

(f) See Figure 4.2 for the sketch of $F_V(v)$:

Problem 4. An optical inspection system is to distinguish among different part types. The probability of a correct classification of any part is 0.98. Suppose that three parts are inspected and that the classifications are independent.

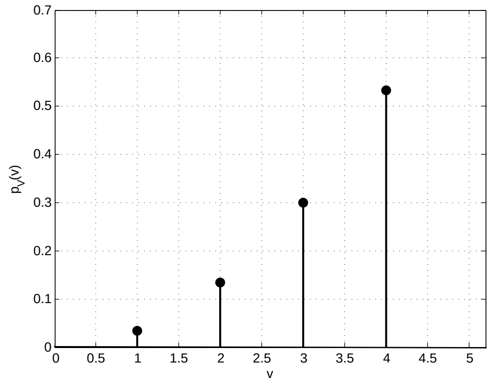

Figure 4.1: Sketch of $p_V(v)$ for Question 3

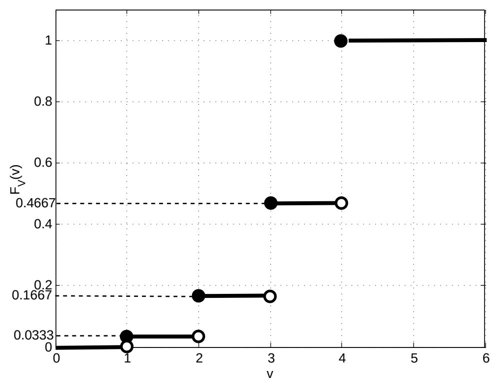

Figure 4.2: Sketch of $F_V(v)$ for Question 3

(a) Let the random variable X denote the number of parts that are correctly classified. Determine the probability mass function of X. [Montgomery and Runger, 2010, Q3-20]

(b) Let the random variable Y denote the number of parts that are incorrectly classified. Determine the probability mass function of Y.

Solution:

(a) X is a binomial random variable with $n = 3$ and $p = 0.98$. Hence,

$$p_X(x) = \begin{cases} \binom{3}{x} 0.98^x (0.02)^{3-x}, & x \in \{0,1,2,3\}, \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

In particular, $p_X(0) = 8 \times 10^{-6}$, $p_X(1) = 0.001176$, $p_X(2) = 0.057624$, and $p_X(3) = 0.941192$. Note that in `MATLAB`, these probabilities can be calculated by evaluating `binopdf(0:3,3,0.98)`.

(b) Y is a binomial random variable with $n = 3$ and $p = 0.02$. Hence,

$$p_Y(y) = \begin{cases} \binom{3}{y} 0.02^y (0.98)^{3-y}, & y \in \{0,1,2,3\}, \\ 0, & \text{otherwise} \end{cases} \tag{4.2}$$

In particular, $p_Y(0) = 0.941192$, $p_Y(1) = 0.057624$, $p_Y(2) = 0.001176$, and $p_Y(3) = 8 \times 10^{-6}$. Note that in `MATLAB`, these probabilities can be calculated by evaluating `binopdf(0:3,3,0.02)`.

Alternatively, note that there are three parts. If X of them are classified correctly, then the number of incorrectly classified parts is $n - X$, which is what we defined as Y. Therefore, $Y = 3 - X$. Hence, $p_Y(y) = P[Y = y] = P[3 - X = y] = P[X = 3 - y] = p_X(3 - y)$.

Problem 5. The thickness of the wood paneling (in inches) that a customer orders is a random variable with the following cdf:

$$F_X(x) = \begin{cases} 0, & x < \frac{1}{8} \\ 0.2, & \frac{1}{8} \le x < \frac{1}{4} \\ 0.9, & \frac{1}{4} \le x < \frac{3}{8} \\ 1 & x \ge \frac{3}{8} \end{cases}$$

Determine the following probabilities:

(a) $P[X \le 1/18]$

(b) $P[X \le 1/4]$

(c) $P[X \le 5/16]$

(d) $P[X > 1/4]$

(e) $P[X \le 1/2]$

[Montgomery and Runger, 2010, Q3-42]

Solution:

(a) $P[X \le 1/18] = F_X(1/18) = 0$ because $\frac{1}{18} < \frac{1}{8}$.

(b) $P[X \le 1/4] = F_X(1/4) = 0.9$

(c) $P[X \le 5/16] = F_X(5/16) = 0.9$ because $\frac{1}{4} < \frac{5}{16} <$ ~~$\frac{1}{8}$~~ $\frac{3}{8}$.

(d) $P[X > 1/4] = 1 - P[X \le 1/4] = 1 - F_X(1/4) = 1 - 0.9 = 0.1$.

(e) $P[X \le 1/2] = F_X(1/2) = 1$ because $\frac{1}{2} > \frac{3}{8}$.

Alternatively, we can also derive the pmf first and then calculate the probabilities.