

HW Solution 3 — Due: Aug 10

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Instructions

- (a) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.
- (b) It is important that you try to solve all problems. (5 pt)
- (c) Late submission will be heavily penalized.
- (d) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 99% of the time.

- (a) What is $P(-|H)$, the conditional probability that a person tests negative given that the person does have the HIV virus?
- (b) What is $P(H|+)$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive?

Solution:

- (a) Because the test is correct 99% of the time,

$$P(-|H) = P(+|H^c) = \boxed{0.01}.$$

- (b) Using Bayes' formula, $P(H|+) = \frac{P(+|H)P(H)}{P(+)}$, where $P(+)$ can be evaluated by the total probability formula:

$$P(+)=P(+|H)P(H)+P(+|H^c)P(H^c)=0.99\times 0.0002+0.01\times 0.9998.$$

Plugging this back into the Bayes' formula gives

$$P(H|+)=\frac{0.99\times 0.0002}{0.99\times 0.0002+0.01\times 0.9998}\approx\boxed{0.0194}.$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 2%. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 2. A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases. A specific design is randomly generated by the Web server when you visit the site. Let A denote the event that the design color is red and let B denote the event that the font size is not the smallest one.

- (a) Use classical probability to evaluate $P(A)$, $P(B)$ and $P(A \cap B)$. Show that the two events A and B are independent by checking whether $P(A \cap B) = P(A)P(B)$.
- (b) Using the values of $P(A)$ and $P(B)$ from the previous part and the fact that $A \perp\!\!\!\perp B$, calculate the following probabilities.
- $P(A \cup B)$
 - $P(A \cup B^c)$
 - $P(A^c \cup B^c)$

[Montgomery and Runger, 2010, Q2-84]

Solution:

- (a) There are

$$|\Omega| = 4 \times 3 \times 5 \times 3 \times 5 \quad (3.1)$$

possible designs. The number of designs whose color is red is given by

$$|A| = 1 \times 3 \times 5 \times 3 \times 5.$$

Note that the “4” in (??) is replaced by “1” because we only consider one color (red). Therefore,

$$P(A) = \frac{1 \times 3 \times 5 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{1}{4}}.$$

Similarly, $|B| = 4 \times 3 \times 4 \times 3 \times 5$ where the “5” in the middle of (??) is replaced by “4” because we can’t use the smallest font size. Therefore,

$$P(B) = \frac{4 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{4}{5}}.$$

For the event $A \cap B$, we replace “4” in (??) by “1” because we need red color and we replace “5” in the middle of (??) by “4” because we can’t use the smallest font size. This gives

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{1 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \frac{1 \times 4}{4 \times 5} = \boxed{\frac{1}{5}} = 0.2.$$

Because $P(A \cap B) = P(A)P(B)$, the events A and B are independent.

(b)

$$(i) P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{4}{5} - \frac{1}{5} = \boxed{\frac{17}{20} = 0.85.}$$

$$(ii) P(A^c \cup B^c) = 1 - P((A^c \cup B^c)^c) = 1 - P(A \cap B) = 1 - 0.2 = \boxed{0.8.}$$

$$(iii) P(A \cup B^c) = 1 - P((A \cup B^c)^c) = 1 - P(A^c \cap B). \text{ Because } A \perp\!\!\!\perp B, \text{ we also have } A^c \perp\!\!\!\perp B. \text{ Hence, } P(A^c \cap B) = P(A^c)P(B) = 1 - \frac{3}{4} \cdot \frac{4}{5} = \frac{2}{5} = \boxed{0.4.}$$

Problem 3. Anne and Betty go fishing. Find the conditional probability that Anne catches no fish given that at least one of them catches no fish. Assume they catch fish independently and that each has probability $0 < p < 1$ of catching no fish. [Gubner, 2006, Q2.62]

Solution: Let A be the event that Anne catches no fish and B be the event that Betty catches no fish. From the question, we know that A and B are independent. The event “at least one of the two women catches nothing” can be represented by $A \cup B$. So we have

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B) - P(A)P(B)} = \frac{p}{2p - p^2} = \boxed{\frac{1}{2 - p}}.$$

Problem 4. Show that if A and B are independent events, then so are A and B^c , A^c and B , and A^c and B^c .

Solution: To show that two events C_1 and C_2 are independent, we need to show that $P(C_1 \cap C_2) = P(C_1)P(C_2)$.

(a) Note that

$$P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B).$$

Because $A \perp\!\!\!\perp B$, the last term can be factored in to $P(A)P(B)$ and hence

$$P(A \cap B^c) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$$

(b) By interchanging the role of A and B in the previous part, we have

$$P(A^c \cap B) = P(B \cap A^c) = P(B)P(A^c).$$

(c) From set theory, we know that $A^c \cap B^c = (A \cup B)^c$. Therefore,

$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B).$$

Because $A \perp\!\!\!\perp B$, we have

$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c). \end{aligned}$$

Remark: By interchanging the roles of A and A^c and/or B and B^c , it follows that if any one of the four pairs is independent, then so are the other three. [Gubner, 2006, p.31]

Problem 5. In this question, each experiment has equiprobable outcomes.

(a) Let $\Omega = \{1, 2, 3, 4\}$, $A_1 = \{1, 2\}$, $A_2 = \{1, 3\}$, $A_3 = \{2, 3\}$.

(i) Determine whether $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.

(ii) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

(iii) Are A_1, A_2 , and A_3 independent?

(b) Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2, 3, 4\}$, $A_2 = A_3 = \{4, 5, 6\}$.

(i) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

(ii) Check whether $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.

(iii) Are A_1, A_2 , and A_3 independent?

Solution:

(a) We have $P(A_i) = \frac{1}{2}$ and $P(A_i \cap A_j) = \frac{1}{4}$.

(i) $P(A_i \cap A_j) = P(A_i)P(A_j)$ for any $i \neq j$.

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$. Hence, $P(A_1 \cap A_2 \cap A_3) = 0$, which is *not* the same as $P(A_1)P(A_2)P(A_3)$.

(iii) No.

(b) We have $P(A_1) = \frac{4}{6} = \frac{2}{3}$ and $P(A_2) = P(A_3) = \frac{3}{6} = \frac{1}{2}$.

(i) $A_1 \cap A_2 \cap A_3 = \{4\}$. Hence, $P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}$.

$$P(A_1)P(A_2)P(A_3) = \frac{2}{3} \frac{1}{2} \frac{1}{2} = \frac{1}{6}.$$

Hence, $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

- (ii) $P(A_2 \cap A_3) = P(A_2) = \frac{1}{2} \neq P(A_2)P(A_3)$
 $P(A_1 \cap A_2) = p(4) = \frac{1}{6} \neq P(A_1)P(A_2)$
 $P(A_1 \cap A_3) = p(4) = \frac{1}{6} \neq P(A_1)P(A_3)$
Hence, $P(A_i \cap A_j) \neq P(A_i)P(A_j)$ for all $i \neq j$.
- (iii) No.

Problem 6 (Majority Voting in Digital Communication). A certain binary communication system has a bit-error rate of 0.1; i.e., in transmitting a single bit, the probability of receiving the bit in error is 0.1. To transmit messages, a three-bit repetition code is used. In other words, to send the message 1, 111 is transmitted, and to send the message 0, 000 is transmitted. At the receiver, if two or more 1s are received, the decoder decides that message 1 was sent; otherwise, i.e., if two or more zeros are received, it decides that message 0 was sent.

Assuming bit errors occur independently, find the probability that the decoder puts out the wrong message.

[Gubner, 2006, Q2.62]

Solution: Let $p = 0.1$ be the bit error rate. Error event \mathcal{E} occurs if there are at least two bit errors. Therefore

$$P(\mathcal{E}) = \binom{3}{2}p^2(1-p) + \binom{3}{3}p^3 = p^2(3-2p).$$

When $p = 0.1$, $P(\mathcal{E}) \approx \boxed{0.028}$.

Problem 7. In an experiment, A , B , C , and D are events with probabilities $P(A \cup B) = \frac{5}{8}$, $P(A) = \frac{3}{8}$, $P(C \cap D) = \frac{1}{3}$, and $P(C) = \frac{1}{2}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find

- (i) $P(A \cap B)$
(ii) $P(B)$
(iii) $P(A \cap B^c)$
(iv) $P(A \cup B^c)$

(b) Are A and B independent?

(c) Find

- (i) $P(D)$
- (ii) $P(C \cap D^c)$
- (iii) $P(C^c \cap D^c)$
- (iv) $P(C|D)$
- (v) $P(C \cup D)$
- (vi) $P(C \cup D^c)$

(d) Are C and D^c independent?

Solution:

(a)

- (i) Because $A \perp B$, we have $A \cap B = \emptyset$ and hence $P(A \cap B) = \boxed{0}$.
- (ii) Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, $P(B) = P(A \cup B) - P(A) + P(A \cap B) = 5/8 - 3/8 + 0 = 2/8 = \boxed{1/4}$.
- (iii) $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) = \boxed{3/8}$.
- (iv) Start with $P(A \cup B^c) = 1 - P(A^c \cap B)$. Now, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) = 1/4$. Hence, $P(A \cup B^c) = 1 - 1/4 = \boxed{3/4}$.

(b) Events A and B are not independent because $P(A \cap B) \neq P(A)P(B)$.

(c)

- (i) Because $C \perp\!\!\!\perp D$, we have $P(C \cap D) = P(C)P(D)$. Hence, $P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1/3}{1/2} = \boxed{2/3}$.
- (ii) $P(C \cap D^c) = P(C) - P(C \cap D) = 1/2 - 1/3 = \boxed{1/6}$.
Alternatively, because $C \perp\!\!\!\perp D$, we know that $C \perp\!\!\!\perp D^c$. Hence, $P(C \cap D^c) = P(C)P(D^c) = \frac{1}{2} \left(1 - \frac{2}{3}\right) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.
- (iii) First, we find $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = 5/6$.
Hence, $P(C^c \cap D^c) = 1 - P(C \cup D) = 1 - 5/6 = \boxed{1/6}$.
Alternatively, because $C \perp\!\!\!\perp D$, we know that $C^c \perp\!\!\!\perp D^c$. Hence, $P(C^c \cap D^c) = P(C^c)P(D^c) = \left(1 - \frac{1}{2}\right) \left(1 - \frac{2}{3}\right) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.
- (iv) Because $C \perp\!\!\!\perp D$, we have $P(C|D) = P(C) = \boxed{1/2}$.
- (v) In part (iii), we already found $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = \boxed{5/6}$.

(vi) $P(C \cup D^c) = 1 - P(C^c \cap D) = 1 - P(C^c)P(D) = 1 - \frac{1}{2} \cdot \frac{2}{3} = \boxed{2/3}$. Note that we use the fact that $C^c \perp\!\!\!\perp D$ to get the second equality.

Alternatively, $P(C \cup D^c) = P(C) + P(D^c) - P(C \cap D^c)$. From (i), we have $P(D) = 2/3$. Hence, $P(D^c) = 1 - 2/3 = 1/3$. From (ii), we have $P(C \cap D^c) = 1/6$. Therefore, $P(C \cup D^c) = 1/2 + 1/3 - 1/6 = 2/3$.

(d) Yes. We know that if $C \perp\!\!\!\perp D$, then $C \perp\!\!\!\perp D^c$.