## A.3 Calculus

**A.15.** ***Integration by parts*** is a technique for simplifying integrals of the form

$$\int a(x)\, b(x) dx.$$

In particular,

$$\int f(x)\, g'(x) dx = f(x)\, g(x) - \int f'(x)\, g(x) dx. \tag{71}$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)dx$ and $dv = g'(x)dx$. Using the Substitution Rule, the integration by parts formula becomes

$$\int u dv = uv - \int v du \tag{72}$$

- The main goal in integration by parts is to choose $u$ and $dv$ to obtain a new integral that is easier to evaluate then the original. In other words, the goal of integration by parts is to go from an integral $\int u dv$ that we dont see how to evaluate to an integral $\int v du$ that we can evaluate.
- Note that when we calculate $v$ from $dv$, we can use *any* of the antiderivative. In other words, we may put in $v + C$ instead of $v$ in (72). Had we included this constant of integration $C$ in (72), it would have eventually dropped out. This is always the case in integration by parts.

For definite integrals, the formula corresponding to (71) is

$$\int_a^b f(x)\, g'(x) dx = f(x)\, g(x)\big|_a^b - \int_a^b f'(x)\, g(x) dx. \tag{73}$$

The corresponding $u$ and $v$ notation is

$$\int_a^b u dv = uv|_a^b - \int_a^b v du \tag{74}$$

It is important to keep in mind that the variables $u$ and $v$ in this formula are functions of $x$ and that the limits of integration in (74) are limits on the variable $x$. Sometimes it is helpful to emphasize this by writing (74) as

$$\int_{x=a}^{b} udv = uv|_{x=a}^{b} - \int_{x=a}^{b} vdu \tag{75}$$

Repeated application of integration by parts gives

$$\int f(x)\,g(x)dx = f(x)\,G_1(x) + \sum_{i=1}^{n-1} (-1)^i f^{(i)}(x)\,G_{i+1}(x) + (-1)^n \int f^{(n)}(x)\,G_n(x)\,dx \tag{76}$$

where $f^{(i)}(x) = \frac{d^i}{dx^i} f(x)$, $G_1(x) = \int g(x)dx$, and $G_{i+1}(x) = \int G_i(x)dx$.

A convenient method for organizing the computations into two columns is called ***tabular integration by parts*** shown in Figure 29 which can be used to derived (76).

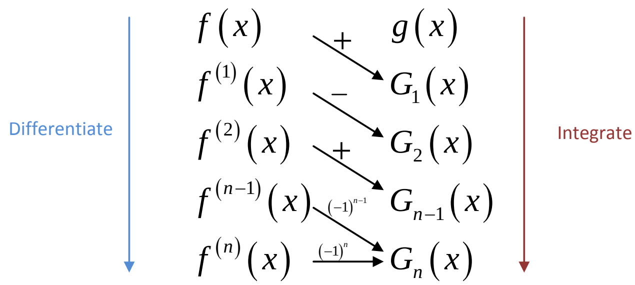


Figure 29: Integration by Parts

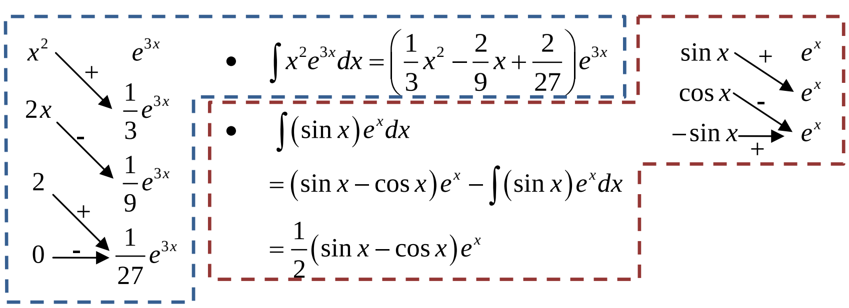


Figure 30: Examples of Integration by Parts using Figure 29.

**Example A.16.** Use ***integration by parts*** to compute the following integrals:

(a) $\int x \ln x dx$.

(b) $\int x^2 e^{-x} dx$.

***Solution***:

(a) $\int x \ln x dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} dx = \frac{x^2}{2} \ln x - \frac{1}{2} \int x dx = \boxed{\frac{x^2}{2} \ln x - \frac{x^2}{4} + C}$

(b) $\int x^2 e^{-x} dx = \left(x^2\right)\left(-e^{-x}\right) - (2x)\left(e^{-x}\right) + (2)\left(-e^{-x}\right) + C = \boxed{-e^{-x}(x^2 + 2x + 2) + C}$.

**A.17.** Integration involving the ***Gaussian function***: There are several important results in probability that are derived from such integrations. It is probably easier to remember or start with the formula for the gaussian pdf because we know that it should integrate to 1:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx = 1. \tag{77}$$

To actually evaluate (prove) such an integral, we simplify it by a change of variable to get an equivalent expression:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x)^2}{2}} dx = 1. \qquad (\star)$$

Even this simplified form is quite tricky to evaluate. The typical procedure is to consider Consider the square of the integral:

$$\left( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x)^2}{2}} dx \right)^2 = \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} e^{-\frac{(x)^2}{2}} dx \right) \left( \int_{-\infty}^{\infty} e^{-\frac{(y)^2}{2}} dy \right).$$

After combining the product on the right into a double integral, we change from Cartesian to polar coordinates. Let $x = r\cos(\theta)$

and $y = r\sin(\theta)$. In which case, $x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$. This gives

$$\begin{aligned}
\left( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right)^2 &= \frac{1}{2\pi} \left( \int_0^{\infty} \int_0^{2\pi} r e^{-\frac{r^2}{2}} d\theta dr \right) \\
&= \frac{1}{2\pi} \left( \int_0^{\infty} r e^{-\frac{r^2}{2}} \left( \int_0^{2\pi} d\theta \right) dr \right) \\
&= \frac{1}{2\pi} \left( 2\pi \int_0^{\infty} r e^{-\frac{r^2}{2}} dr \right) = -\, e^{-\frac{r^2}{2}} \Big|_0^{\infty} = 1
\end{aligned}$$

which complete the proof.

Now that we have derive (77), it can then be used to show several important results some of which are provided below.

**Example A.18.** Analytically derive the following facts:

(a) $\int\limits_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$ for $\alpha > 0$.

(b) $\int\limits_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0$

Remark: This shows that $\mathbb{E}[X] = 0$ when $X \sim \mathcal{N}(0, 1)$.

(c) $\int\limits_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$.

Hint: Write $x^2 e^{-\frac{x^2}{2}}$ as

$$x^2 e^{-\frac{x^2}{2}} = x \times x e^{-\frac{x^2}{2}}$$

and use integration by parts.

Remark: This shows that $\mathbb{E}\left[X^2\right] = \operatorname{Var} X = 1$ when $X \sim \mathcal{N}(0, 1)$.

(d) $\int\limits_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$ for $\alpha > 0$.

(e) $\int\limits_0^\infty x^2 e^{-\alpha x^2} dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$ for $\alpha > 0$.

(f) $\int\limits_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = e^{\frac{1}{2}s^2}$.

Hint: Completing the square: $x^2 - 2sx = (x-s)^2 - s^2$.

Remark: This shows that when $X \sim \mathcal{N}(0,1)$,

$$\mathbb{E}\left[e^{sX}\right] = e^{\frac{1}{2}s^2}.$$

To find the Fourier transform of $f_X$, simply substitute $s = -j\omega = -j2\pi f$ to get

$$\mathbb{E}\left[e^{-j\omega X}\right] = e^{-\frac{1}{2}\omega^2} = e^{-2\pi^2 f^2}.$$

To find characteristic function of the standard Gaussian $X$, we substitute $s = jt$ to get

$$\varphi_X(t) = \mathbb{E}\left[e^{jtX}\right] = e^{-\frac{1}{2}t^2}.$$

(g) When $X \sim \mathcal{N}(m, \sigma^2)$,

(i) $\mathbb{E}\left[e^{sX}\right] = e^{sm + \frac{1}{2}s^2\sigma^2}$.

(ii) Fourier transform:

$$\mathcal{F}\{f_X\} = \int\limits_{-\infty}^{\infty} f_X(x)\, e^{-j\omega x} dx = e^{-j\omega m - \frac{1}{2}\omega^2\sigma^2} = e^{-j2\pi f m - 2\pi^2 f^2 \sigma^2}.$$

(iii) Characteristic function:

$$\varphi_X(t) = \mathbb{E}\left[e^{jtX}\right] = e^{jtm - \frac{1}{2}t^2\sigma^2}$$

***Solution***:

(a) Let $y = \sqrt{2\alpha}x$. Then, $\frac{1}{2}y^2 = \alpha x^2$ and $dx = \frac{1}{\sqrt{2\alpha}} dy$. Hence,

$$\int\limits_{-\infty}^{\infty} e^{-\alpha x^2} dx = \frac{1}{\sqrt{2\alpha}} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy.$$

see (*) above

We have already shown ~~in part (a)~~ that $\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = 1$.

Hence, $\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy = \sqrt{2\pi}$ and

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy = \frac{\sqrt{2\pi}}{\sqrt{2\alpha}} = \sqrt{\frac{\pi}{\alpha}}.$$

(b) $xe^{-\frac{(x)^2}{2}}$ is an odd function.

(c) Use integration by parts: separating

$$x^2 e^{-\frac{x^2}{2}} = x \times x e^{-\frac{x^2}{2}}$$

to get

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx = -x e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

(d) Let $y = \sqrt{2\alpha}x$. Then, $\frac{1}{2}y^2 = \alpha x^2$ and $dx = \frac{1}{\sqrt{2\alpha}} dy$. Hence,

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx &= \int_{-\infty}^{\infty} \frac{1}{2\alpha} y^2 e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{2\alpha}} dy \\ &= \frac{\sqrt{2\pi}}{(2\alpha)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} y^2 e^{-\frac{1}{2}y^2} dy \\ &= \frac{\sqrt{2\pi}}{(2\alpha)^{\frac{3}{2}}} \end{aligned}$$

(e) $x^2 e^{-\alpha x^2}$ is an even function. Hence,

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = 2 \int_{0}^{\infty} x^2 e^{-\alpha x^2} dx.$$

(f) Applying the hint, we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{sx} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x^2-2sx+s^2\right)} e^{\frac{1}{2}s^2} dx$$
$$= e^{\frac{1}{2}s^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-s)^2} dx = e^{\frac{1}{2}s^2}$$

(g) For $X \sim \mathcal{N}(m, \sigma^2)$, we have

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}.$$

(i)

$$\mathbb{E}\left[e^{sX}\right] = \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$
$$= \int_{-\infty}^{\infty} e^{s(\sigma y+m)} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}y^2} \sigma dy$$
$$= e^{sm} \int_{-\infty}^{\infty} e^{s\sigma y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = e^{sm} e^{\frac{1}{2}(s\sigma)^2}$$
$$= e^{sm+\frac{1}{2}s^2\sigma^2}$$

(ii) To find the Fourier transform of $f_X$, simply substitute $s = -j\omega = -j2\pi f$ into the answer from part (g.i).

(iii) To find characteristic function of the standard Gaussian $X$, we substitute $s = jt$ into the answer from part (g.i).

**A.19** (Differential of integral)**.** ***Leibniz's Rule***: Let $g : \mathbb{R}^2 \to \mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}$, and $b : \mathbb{R} \to \mathbb{R}$ be $C^1$. Then $f(x) = \int\limits_{a(x)}^{b(x)} g(x, y)dy$ is $C^1$ and

$$\boxed{f'(x) = b'(x)\, g(x, b(x)) - a'(x)\, g(x, a(x)) + \int\limits_{a(x)}^{b(x)} \frac{\partial g}{\partial x}(x, y)dy.} \tag{78}$$

In particular, we have

$$\frac{d}{dx}\int\limits_{a}^{x} f(t)dt = f(x), \tag{79}$$

$$\frac{d}{dx}\int\limits_{a}^{v(x)} f(t)dt = \frac{dv}{dx}\frac{d}{dv}\int\limits_{a}^{v(x)} f(t)dt = f(v(x))\, v'(x), \tag{80}$$

$$\frac{d}{dx}\int\limits_{u(x)}^{v(x)} f(t)dt = \frac{d}{dx}\left(\int\limits_{a}^{v(x)} f(t)dt - \int\limits_{a}^{u(x)} f(t)dt\right) \tag{81}$$

$$= f(v(x))\, v'(x) - f(u(x))\, u'(x). \tag{82}$$