Sirindhorn International Institute of Technology
Thammasat University
School of Information, Computer and Communication Technology

ECS315 2012/1 Part IV.2 Dr.Prapun

11.5 Function of Continuous Random Variables: SISO

Reconsider the derived random variable $Y = g(X)$.

Recall that we can find $\mathbb{E}Y$ easily by (27):

$$\mathbb{E}Y = \mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) f_X(x) dx.$$

However, there are cases when we have to evaluate probability directly involving the random variable Y or find $f_Y(y)$ directly.

Recall that for discrete random variables, it is easy to find $p_Y(y)$ by adding all $p_X(x)$ over all x such that $g(x) = y$:

$$p_Y(y) = \sum_{x: g(x) = y} p_X(x). \tag{28}$$

For continuous random variables, it turns out that we can't[44] simply integrate the pdf of X to get the pdf of Y.

11.55. For $Y = g(X)$, if you want to find $f_Y(y)$, the following ***two-step procedure*** will always work and is easy to remember:

(a) Find the cdf $F_Y(y) = P[Y \le y]$.

(b) Compute the pdf from the cdf by "finding the derivative" $f_Y(y) = \frac{d}{dy} F_Y(y)$ (as described in 11.13).

[44] When you applied Equation (28) to continuous random variables, what you would get is $0 = 0$, which is true but not interesting nor useful.

11.56. *Linear Transformation*: Suppose $Y = aX + b$. Then, the cdf of Y is given by

$$F_Y(y) = P[Y \le y] = P[aX + b \le y] = \begin{cases} P\left[X \le \frac{y-b}{a}\right], & a > 0, \\ P\left[X \ge \frac{y-b}{a}\right], & a < 0. \end{cases}$$

Now, by definition, we know that

$$P\left[X \le \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right),$$

and

$$\begin{aligned} P\left[X \ge \frac{y-b}{a}\right] &= P\left[X > \frac{y-b}{a}\right] + P\left[X = \frac{y-b}{a}\right] \\ &= 1 - F_X\left(\frac{y-b}{a}\right) + P\left[X = \frac{y-b}{a}\right]. \end{aligned}$$

For continuous random variable, $P\left[X = \frac{y-b}{a}\right] = 0$. Hence,

$$\boxed{F_Y(y) = \begin{cases} F_X\left(\frac{y-b}{a}\right), & a > 0, \\ 1 - F_X\left(\frac{y-b}{a}\right), & a < 0. \end{cases}}$$

Finally, fundamental theorem of calculus and chain rule gives

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} \frac{1}{a}f_X\left(\frac{y-b}{a}\right), & a > 0, \\ -\frac{1}{a}f_X\left(\frac{y-b}{a}\right), & a < 0. \end{cases}$$

Note that we can further simplify the final formula by using the $|\cdot|$ function:

$$\boxed{f_Y(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right), \quad a \ne 0.} \tag{29}$$

Graphically, to get the plots of f_Y, we compress f_X horizontally by a factor of a, scale it vertically by a factor of $1/|a|$, and shift it to the right by b.

Of course, if $a = 0$, then we get the uninteresting degenerated random variable $Y \equiv b$.

Example 11.57. Amplitude modulation in certain communication systems can be accomplished using various nonlinear devices such as a semiconductor diode. Suppose we model the nonlinear device by the function $Y = X^2$. If the input X is a continuous random variable, find the density of the output $Y = X^2$.

Exercise 11.58. Suppose $X \sim \mathcal{N}(m, \sigma^2)$ and $Y = aX + b$ for some constants a and b. Use (29) to show that $X \sim \mathcal{N}(am + b, a^2\sigma^2)$.

Exercise 11.59 (F2011)**.** Suppose X is uniformly distributed on the interval $(1, 2)$. ($X \sim \mathcal{U}(1, 2)$.) Let $Y = \frac{1}{X^2}$.

(a) Find $f_Y(y)$.

(b) Find $\mathbb{E}Y$.

Exercise 11.60 (F2011)**.** Consider the function

$$g(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Suppose $Y = g(X)$, where $X \sim \mathcal{U}(-2, 2)$.

Remark: The function g operates like a **full-wave rectifier** in that if a positive input voltage X is applied, the output is $Y = X$, while if a negative input voltage X is applied, the output is $Y = -X$.

(a) Find $\mathbb{E}Y$.

(b) Plot the cdf of Y.

(c) Find the pdf of Y

	Discrete	Continuous
$P[X \in B] =$	$\sum_{x \in B} p_X(x)$	$\int_B f_X(x)dx$
$P[X = x] =$	$p_X(x) = F(x) - F(x^-)$	0
Interval prob.	$P^X((a,b]) = F(b) - F(a)$ $P^X([a,b]) = F(b) - F(a^-)$ $P^X([a,b)) = F(b^-) - F(a^-)$ $P^X((a,b)) = F(b^-) - F(a)$	$P^X((a,b]) = P^X([a,b])$ $= P^X([a,b)) = P^X((a,b))$ $= \int_a^b f_X(x)dx = F(b) - F(a)$
$\mathbb{E}X =$	$\sum_x x p_X(x)$	$\int_{-\infty}^{+\infty} x f_X(x)dx$
For $Y = g(X)$,	$p_Y(y) = \sum_{x: g(x)=y} p_X(x)$	$f_Y(y) = \frac{d}{dy} P[g(X) \le y]$. Alternatively, $f_Y(y) = \sum_k \frac{f_X(x_k)}{\lvert g'(x_k)\rvert}$, x_k are the real-valued roots of the equation $y = g(x)$.
For $Y = g(X)$, $P[Y \in B] =$	$\sum_{x: g(x) \in B} p_X(x)$	$\int_{\{x: g(x) \in B\}} f_X(x)dx$
$\mathbb{E}[g(X)] =$	$\sum_x g(x) p_X(x)$	$\int_{-\infty}^{+\infty} g(x) f_X(x)dx$
$\mathbb{E}[X^2] =$	$\sum_x x^2 p_X(x)$	$\int_{-\infty}^{+\infty} x^2 f_X(x)dx$
$\operatorname{Var} X =$	$\sum_x (x - \mathbb{E}X)^2 p_X(x)$	$\int_{-\infty}^{+\infty} (x - \mathbb{E}X)^2 f_X(x)dx$

Table 5: Important Formulas for Discrete and Continuous Random Variables

11.6 Pairs of Continuous Random Variables

In this section, we start to look at more than one continuous random variables. You should find that many of the concepts and formulas are similar if not the same as the ones for pairs of ***discrete*** random variables which we have already studied. For discrete random variables, we use summations. Here, for continuous random variables, we use integrations.

Recall that for a pair of discrete random variables, the joint pmf $p_{X,Y}(x,y)$ completely characterizes the probability model of two random variables X and Y. In particular, it does not only capture the probability of X and probability of Y individually, it also capture the relationship between them. For continuous random variable, we replace the joint pmf by joint pdf.

Definition 11.61. We say that two random variables X and Y are ***jointly continuous*** with ***joint pdf*** $f_{X,Y}(x,y)$ if[45] for any **region** R on the (x,y) plane

$$P[(X,Y) \in R] = \iint\limits_{\{(x,y):(x,y)\in R\}} f_{X,Y}(x,y)dxdy \tag{30}$$

To understand where Definition 11.61 comes from, it is helpful to take a careful look at Table 6.

	Discrete	Continuous
$P[X \in B]$	$\sum\limits_{x \in B} p_X(x)$	$\int\limits_B f_X(x)dx$
$P[(X,Y) \in R]$	$\sum\limits_{(x,y):(x,y)\in R} p_{X,Y}(x,y)$	$\iint\limits_{\{(x,y):(x,y)\in R\}} f_{X,Y}(x,y)dxdy$

Table 6: pmf vs. pdf

For us, Definition 11.61 is ***useful*** because if you know that a function $f(x,y)$ is a joint pdf of a pair of random variables, then

[45] Remark: If you want to check that a function $f(x,y)$ is the joint pdf of a pair of random variables (X,Y) by using the above definition, you will need to check that (30) is true for any region R. This is not an easy task. Hence, we do not usually use this definition for such kind of test. There are some mathematical facts that can be derived from this definition. Such facts produce easier condition(s) than (30) but we will not talk about them here.

you can ***calculate*** countless possibilities of probabilities involving these two random variables via (30). (See, e.g. Example 11.64.) However, the actual calculation of probability from (30) can be difficult if you have non-rectangular region R or if you have a complicated joint pdf. In other words, the formula itself is straightforward and simple, but to carry it out may require that you review some multi-variable integration technique from your calculus class.

Note also that the joint pdf's definition extends the interpretation/approximation that we previously discussed for one random variable. Recall that for a single random variable, the pdf is a measure of ***probability per unit length***. In particular, if you want to find the probability that the value of a random variable X falls inside some small interval, say the interval $[1.5, 1.6]$, then this probability can be approximated by

$$P[1.5 \le X \le 1.6] \approx f_X(1.5) \times 0.1.$$

More generally, for small value of interval length d, the probability that the value of X falls within a small interval $[x, x+d]$ can be approximated by

$$P[x \le X \le x+d] \approx f_X(x) \times d. \tag{31}$$

Usually, instead of using d, we use Δx and hence

$$P[x \le X \le x+\Delta x] \approx f_X(x) \times \Delta x. \tag{32}$$

11.62. Intuition/Approximation: For two random variables X and Y, the joint pdf $f_{X,Y}(x,y)$ measures ***probability per unit area***:

$$P[x \le X \le x+\Delta x\,,\ y \le Y \le y+\Delta y] \approx f_{X,Y}(x,y) \times \Delta x \times \Delta y. \tag{33}$$

Do not forget that the comma signifies the "and" (intersection) operation.

11.63. There are two important characterizing properties of joint pdf:

(a) $f_{X,Y} \geq 0$ a.e.

(b) $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x, y)dxdy = 1$

Example 11.64. Consider a probability model of a pair of random variables uniformly distributed over a rectangle in the X-Y plane:

$$f_{X,Y}(x,y) = \begin{cases} c, & 0 \leq x \leq 5, 0 \leq y \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

Find the constant c, $P[2 \leq X \leq 3, 1 \leq Y \leq 3]$, and $P[Y > X]$

11.65. Other important properties and definitions for a pair of continuous random variables are summarized in Table 7 along with their "discrete counterparts".

	Discrete	Continuous
$P[(X,Y) \in R]$	$\sum\limits_{(x,y):(x,y)\in R} p_{X,Y}(x,y)$	$\iint\limits_{\{(x,y):(x,y)\in R\}} f_{X,Y}(x,y)dxdy$
Joint to Marginal:	$p_X(x) = \sum\limits_{y} p_{X,Y}(x,y)$	$f_X(x) = \int\limits_{-\infty}^{+\infty} f_{X,Y}(x,y)dy$
(Law of Total Prob.)	$p_Y(y) = \sum\limits_{x} p_{X,Y}(x,y)$	$f_Y(y) = \int\limits_{-\infty}^{+\infty} f_{X,Y}(x,y)dx$
$P[X > Y]$	$\sum\limits_{x}\sum\limits_{y:y<x} p_{X,Y}(x,y)$	$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{x} f_{X,Y}(x,y)dydx$
	$= \sum\limits_{y}\sum\limits_{x:x>y} p_{X,Y}(x,y)$	$= \int\limits_{-\infty}^{+\infty}\int\limits_{y}^{\infty} f_{X,Y}(x,y)dxdy$
$P[X = Y]$	$\sum\limits_{x} p_{X,Y}(x,x)$	0
$X \perp\!\!\!\perp Y$	$p_{X,Y}(x,y) = p_X(x)p_Y(y)$	$f_{X,Y}(x,y) = f_X(x)f_Y(y)$
Conditional	$p_{X\mid Y}(x\mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\mid Y}(x\mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

Table 7: Important formulas for a pair of discrete RVs and a pair of Continuous RVs

Exercise 11.66 (F2011)**.** Random variables X and Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Check that $c = 2$.

(b) In Figure 18, specify the region of nonzero pdf.

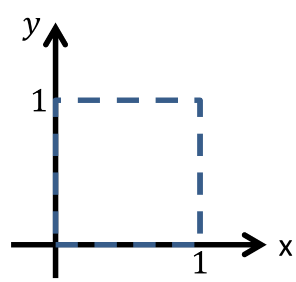

Figure 18: Figure for Exercise 11.66b.

(c) Find the marginal density $f_X(x)$.

(d) Check that $\mathbb{E}X = \frac{2}{3}$.

(e) Find the marginal density $f_Y(y)$.

(f) Find $\mathbb{E}Y$

Definition 11.67. The ***joint cumulative distribution function*** (***joint cdf***) of random variables X and Y (of any type(s)) is defined as

$$F_{X,Y}(x, y) = P[X \le x, Y \le y].$$

- Although its definition is simple, we rarely use the joint cdf to study probability models. It is easier to work with a probability mass function when the random variables are discrete, or a probability density function if they are continuous.

11.68. The joint cdf for a pair of random variables (of any type(s)) has the following properties[46].:

(a) $0 \le F_{X,Y}(x, y) \le 1$

  (i) $F_{X,Y}(\infty, \infty) = 1$.

  (ii) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$.

(b) Joint to Marginal: $F_X(x) = F_{X,Y}(x, \infty)$ and $F_Y(y) = F_{X,Y}(\infty, y)$. In words, we obtain the marginal cdf F_X and F_Y directly from $F_{X,Y}$ by setting the unwanted variable to ∞.

(c) If $x_1 \le x_2$ and $y_1 \le y_2$, then $F_{X,Y}(x_1, y_1) \le F_{X,Y}(x_2, y_2)$

11.69. The joint cdf for a pair of ***continuous*** random variables also has the following properties:

(a) $F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v)\,dv\,du$.

(b) $f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$.

[46]Note that when we write $F_{X;Y}(x, \infty)$, we mean $\lim_{y \to \infty} F_{X;Y}(x, y)$. Similar limiting definition applies to $F_{X;Y}(\infty, \infty)$, $F_{X;Y}(-\infty, y)$, $F_{X;Y}(x, -\infty)$, and $F_{X;Y}(\infty, y)$

11.70. ***Independence***:

The following statements are equivalent:

(a) Random variables X and Y are ***independent***.

(b) $[X \in B] \perp\!\!\!\perp [Y \in C]$ for all B, C.

(c) $P[X \in B, Y \in C] = P[X \in B] \times P[Y \in C]$ for all B, C.

(d) $f_{X,Y}(x, y) = f_X(x) \times f_Y(y)$ for all x, y.

(e) $F_{X,Y}(x, y) = F_X(x) \times F_Y(y)$ for all x, y.

Exercise 11.71 (F2011)**.** Let X_1 and X_2 be i.i.d. $\mathcal{E}(1)$

(a) Find $P[X_1 = X_2]$.

(b) Find $P\left[X_1^2 + X_2^2 = 13\right]$.

11.7 Function of a Pair of Continuous Random Variables: MISO

There are many situations in which we observe two random variables and use their values to compute a new random variable.

Example 11.72. Signal in ***additive noise***: When we says that a random signal X is transmitted over a channel subject to additive noise N, we mean that at the receiver, the received signal Y will be $X+N$. Usually, the noise is assumed to be zero-mean Gaussian noise; that is $N \sim \mathcal{N}(0, \sigma_N^2)$ for some noise power σ_N^2.

Example 11.73. In a ***wireless channel***, the transmitted signal X is corrupted by fading (multiplicative noise). More specifically, the received signal Y at the receiver's antenna is $Y = H \times X$.

Remark: In the actual situation, the signal is further corrupted by additive noise N and hence $Y = HX + N$. However, this expression for Y involves more than two random variables and hence we we will not consider it here.

	Discrete	Continuous
$\mathbb{E}[Z]$	$\sum_{x}\sum_{y} g(x,y)p_{X;Y}(x,y)$	$\int_{1}^{+1}\int_{1}^{+1} g(x,y)f_{X;Y}(x,y)dxdy$
$P[Z \in B]$	$\sum_{(x;y):\,g(x;y)2B} p_{X;Y}(x,y)$	$\iint_{f(x;y):\,g(x;y)2Bg} f_{X;Y}(x,y)dxdy$
$Z = X + Y$	$p_Z(z) = \sum_{x} p_{X;Y}(x, z-x)$ $= \sum_{y} p_{X;Y}(z-y, y)$	$f_Z(z) = \int_{1}^{+1} f_{X;Y}(x, z-x)dx$ $= \int_{1}^{+1} f_{X;Y}(z-y, y)dy$
$X \perp\!\!\!\perp Y$	$p_{X+Y} = p_X * p_Y$	$f_{X+Y} = f_X * f_Y$

Table 8: Important formulas for function of a pair of RVs. Unless stated otherwise, the function is defined as $Z = g(X, Y)$

11.74. Consider a new random variable Z defined by

$$Z = g(X, Y).$$

Table 8 summarizes the basic formulas involving this derived random variable.

11.75. When X and Y are continuous random variables, it may be of interest to find the pdf of the derived random variable $Z = g(X, Y)$. It is usually helpful to devide this task into two steps:

(a) Find the cdf $F_Z(z) = P[Z \le z] = \iint_{g(x,y)\le z} f_{X,Y}(x,y)dxdy$

(b) $f_W(w) = \frac{d}{dw}F_W(w)$.

Example 11.76. Suppose X and Y are i.i.d. $\mathcal{E}(3)$. Find the pdf of $W = Y/X$.

Exercise 11.77 (F2011)**.** Let X_1 and X_2 be i.i.d. $\mathcal{E}(1)$.

(a) Define $Y = \min\{X_1, X_2\}$. (For example, when $X_1 = 6$ and $X_2 = 4$, we have $Y = 4$.) Describe the random variable Y. Does it belong to any known family of random variables? If so, what is/are its parameters?

(b) Define $Y = \min\{X_1, X_2\}$ and $Z = \max\{X_1, X_2\}$. Find $f_{Y,Z}(2, 1)$.

(c) Define $Y = \min\{X_1, X_2\}$ and $Z = \max\{X_1, X_2\}$. Find $f_{Y,Z}(1, 2)$.

11.78. Observe that finding the pdf of $Z = g(X, Y)$ is a time-consuming task. If you goal is to find $\mathbb{E}[Z]$ do not forget that it can be calculated directly from

$$\mathbb{E}[g(X,Y)] = \int\int g(x,y) f_{X,Y}(x,y)\,dx\,dy.$$

11.79. The following property is valid for any kind of random variables:

$$\mathbb{E}\left[\sum_i Z_i\right] = \sum_i \mathbb{E}[Z_i].$$

Furthermore,

$$\mathbb{E}\left[\sum_i g_i(X,Y)\right] = \sum_i \mathbb{E}[g_i(X,Y)].$$

	Discrete	Continuous
$P[X \in B]$	$\sum_{x \in B} p_X(x)$	$\int_B f_X(x)dx$
$P[(X,Y) \in R]$	$\sum_{(x,y):(x,y)\in R} p_{X,Y}(x,y)$	$\iint_{\{(x,y):(x,y)\in R\}} f_{X,Y}(x,y)dxdy$
Joint to Marginal:	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy$
(Law of Total Prob.)	$p_Y(y) = \sum_x p_{X,Y}(x,y)$	$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dx$
$P[X > Y]$	$\sum_x \sum_{y:y<x} p_{X,Y}(x,y)$ $= \sum_y \sum_{x:x>y} p_{X,Y}(x,y)$	$\int_{-\infty}^{+\infty} \int_{-\infty}^{x} f_{X,Y}(x,y)dydx$ $= \int_{-\infty}^{+\infty} \int_{y}^{\infty} f_{X,Y}(x,y)dxdy$
$P[X = Y]$	$\sum_x p_{X,Y}(x,x)$	0
$X \perp\!\!\!\perp Y$	$p_{X,Y}(x,y) = p_X(x)p_Y(y)$	$f_{X,Y}(x,y) = f_X(x)f_Y(y)$
Conditional	$p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
$\mathbb{E}[g(X,Y)]$	$\sum_x \sum_y g(x,y)p_{X,Y}(x,y)$	$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y)f_{X,Y}(x,y)dxdy$
$P[g(X,Y) \in B]$	$\sum_{(x,y):g(x,y)\in B} p_{X,Y}(x,y)$	$\iint_{\{(x,y):g(x,y)\in B\}} f_{X,Y}(x,y)dxdy$
$Z = X + Y$	$p_Z(z) = \sum_x p_{X,Y}(x,z-x)$ $= \sum_y p_{X,Y}(z-y,y)$	$f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(x,z-x)dx$ $= \int_{-\infty}^{+\infty} f_{X,Y}(z-y,y)$

Table 9: pmf vs. pdf

11.80. ***Independence***: At this point, it is useful to summarize what we know about independence. The following statements are equivalent:

(a) Random variables X and Y are ***independent***.

(b) $[X \in B] \perp\!\!\!\perp [Y \in C]$ for all B, C.

(c) $P[X \in B, Y \in C] = P[X \in B] \times P[Y \in C]$ for all B, C.

(d) For discrete RVs, $p_{X,Y}(x,y) = p_X(x) \times p_Y(y)$ for all x, y.
For continuous RVs, $f_{X,Y}(x,y) = f_X(x) \times f_Y(y)$ for all x, y.

(e) $F_{X,Y}(x,y) = F_X(x) \times F_Y(y)$ for all x, y.

(f) $\mathbb{E}[h(X)g(Y)] = \mathbb{E}[h(X)]\,\mathbb{E}[g(Y)]$ for all functions h and g.

Definition 11.81. All of the definitions involving expectation of a function of two random variables are the same as in the discrete case:

- **Correlation** between X and Y: $\mathbb{E}[XY]$.
- **Covariance** between X and Y:

$$\operatorname{Cov}[X,Y] = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] = \mathbb{E}[XY] - \mathbb{E}X\mathbb{E}Y.$$

- $\operatorname{Var} X = \operatorname{Cov}[X,X]$.
- X and Y are said to be ***uncorrelated*** if and only if $\operatorname{Cov}[X,Y] = 0$.
- X and Y are said to be ***orthogonal*** if $\mathbb{E}[XY] = 0$.
- ***Correlation coefficient***: $\rho_{XY} = \frac{\operatorname{Cov}[X,Y]}{\sigma_X \sigma_Y}$

Exercise 11.82 (F2011)**.** Continue from Exercise 11.66. We found that the joint pdf is given by

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Also recall that $\mathbb{E}X = \frac{2}{3}$ and $\mathbb{E}Y = \frac{1}{3}$.

(a) Find $\mathbb{E}[XY]$

(b) Are X and Y uncorrelated?

(c) Are X and Y independent?

Example 11.83. The ***bivariate Gaussian*** or ***bivariate normal density*** is a generalization of the univariate $\mathcal{N}(m,\sigma^2)$ density. For bivariate normal, $f_{X,Y}(x,y)$ is

$$\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{ -\frac{\left(\frac{x-\mathbb{E}X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mathbb{E}X}{\sigma_X}\right)\left(\frac{y-\mathbb{E}Y}{\sigma_Y}\right) + \left(\frac{y-\mathbb{E}Y}{\sigma_Y}\right)^2}{2(1-\rho^2)} \right\}. \tag{34}$$

Important properties:

(a) $\rho = \frac{\operatorname{Cov}[X,Y]}{\sigma_X\sigma_Y} \in (-1,1)$ [26, Thm. 4.31]

(b) [26, Thm. 4.28]

(c) $X \perp\!\!\!\perp Y$ is equivalent to "X and Y are uncorrelated."

Figure 19: Samples from bivariate Gaussian distributions.

Figure 20: Effect of ρ on bivariate Gaussian distribution. Note that the marginal pdfs for both X and Y are all standard Gaussian.