

EXAM 1 — Name

ID

*Lecturer: Prapun Suksompong, Ph.D.***Instructions**

- (a) Including this cover page, there are 12 pages. The last page is the formula sheet.
- (b) Read these instructions and the questions carefully.
- (c) Closed book. Closed notes.
- (d) Basic calculators, e.g. FX-991MS, are permitted, but borrowing is not allowed.
- (e) Allocate your time wisely. Some easy questions give many points.
- (f) The use of communication devices including mobile phones is prohibited in the examination room. Put all of your communication devices in your bag and leave it at the front of the examination room.
- (g) Do not forget to write your first name and the last three digits of your ID in the spaces provide on the top of each examination page, starting from page 2.
- (h) Write down all the steps that you have done to obtain your answers. You may not get any credit even when your final answer is correct without showing how you get your answer.
- (i) Some points are reserved for reducing answers into their simplest forms.
- (j) Do not cheat.
- (k) Do not panic.

Problem 1. (18 pt) In an experiment, A , B , C , and D are events with probabilities $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{8}$, $P(C) = \frac{5}{8}$, and $P(D) = \frac{3}{8}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find

(i) (2 pt) $P(A \cap B)$

(ii) (2 pt) $P(A \cup B)$

(iii) (2 pt) $P(A \cap B^c)$

(iv) (2 pt) $P(A \cup B^c)$

(b) (1 pt) Are A and B independent?

(c) (2 pt) Note that $P(C) + P(D) = 1$. Does this mean $D = C^c$? Justify your answer.

(d) Find

(i) (2 pt) $P(C \cap D)$

(ii) (2 pt) $P(C \cap D^c)$

(iii) (2 pt) $P(C^c \cap D^c)$

(e) (1 pt) Are C^c and D^c independent?

Solution:

(a)

(i) Because A and B are disjoint, we have $A \cap B = \emptyset$ and hence $P(A \cap B) = P(\emptyset) = \boxed{0}$.

(ii) Because $P(A \cap B) = \boxed{0}$, $P(A \cup B) = P(A) + P(B) = \boxed{3/8} \approx 0.375$.

(iii) $P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B) = P(A) = \boxed{1/4} \approx 0.25$.

(iv) Start with $P(A \cup B^c) = 1 - P((A \cup B^c)^c) = 1 - P(A^c \cap B)$. Now, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) = 1/8$. Hence, $P(A \cup B^c) = 1 - 1/8 = \boxed{7/8} \approx 0.875$.

Alternatively, note that we can express $A \cup B^c$ as a disjoint union $B^c \cup (A \cap B)$. Hence, $P(A \cup B^c) = P(B^c) + P(A \cap B) = P(B^c) + 0 = 1 - P(B) = 1 - \frac{1}{8} = \frac{7}{8}$.

(b) To test independence, we compare $P(A \cap B)$ and $P(A)P(B)$. From part (a), we know that $P(A \cap B) = 0$. We can also evaluate $P(A)P(B) = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32} \neq 0$.

Events A and B are not independent because $P(A \cap B) \neq P(A)P(B)$.

- (c) Suppose $D = C^c$ as we are asked to test. Then, $C \cap D = C \cap C^c = \emptyset$ which implies $P(C \cap D) = P(\emptyset) = 0$. However, in this question, we are given that $C \perp\!\!\!\perp D$ which implies $P(C \cap D) = P(C)P(D)$. To have $P(C \cap D) = 0$ as well, we would need $P(C) = 0$ or $P(D) = 0$, none of which is true for this question. So, $\boxed{D \neq C^c}$.

Remarks:

- If $D = C^c$, we have $C \cup D = C \cup C^c = \Omega$ where the unions are disjoint unions. Hence, $P(C \cup D) = P(C) + P(D)$ which in turn the same as $P(\Omega) = 1$. Therefore, we can conclude that $P(D) + P(C) = 1$. This is NOT what you are asked to test in this question.
- It is tempting, after being given the fact that $P(D) + P(C) = 1$, to move $P(C)$ to the RHS and get $P(D) = 1 - P(C) = P(C^c)$. There is nothing wrong so far. However, you can not go further and conclude that $D = C^c$. Simply having two sets with the same probability does not mean the two sets are the same!
- It is possible to construct C and D such that $P(C) + P(D) = 1$, $D = C^c$ and $C \perp\!\!\!\perp D$. For example, try setting $D = \Omega$ and $C = \emptyset$. This cannot be done in our question though because we are given $P(C)$ which is non-zero.

(d)

- (i) Because C and D are independent, $P(C \cap D) = P(C)P(D) = \frac{5}{8} \frac{3}{8} = \boxed{15/64} \approx 0.234$.

- (ii) $P(C \cap D^c) = P(C \setminus D) = P(C) - P(C \cap D) = 5/8 - 15/64 = \boxed{25/64} \approx 0.391$.

Alternatively, we know that if C and D are independent, then so are C and D^c . Therefore, $P(C \cap D^c) = P(C)P(D^c) = P(C)(1 - P(D)) = \frac{5}{8} (1 - \frac{3}{8}) = \frac{25}{64}$.

- (iii) From Venn diagrams, we see that $C \cup D$ can be expressed as a disjoint union $(C \cap D^c) \cup D$. Therefore, $P(C \cup D) = P(C \cap D^c) + P(D) = 25/64 + 3/8 = 49/64$. Hence, $P(C^c \cap D^c) = 1 - P(C \cup D) = 1 - 49/64 = \boxed{15/64} \approx 0.234$.

Alternatively, we know that if C and D are independent, then so are C^c and D^c . Therefore, $P(C^c \cap D^c) = P(C^c)P(D^c) = (1 - P(C))(1 - P(D)) = (1 - \frac{5}{8})(1 - \frac{3}{8}) = \frac{15}{64}$.

- (e) $\boxed{\text{Yes.}}$ We know that if $C \perp\!\!\!\perp D$, then $C^c \perp\!\!\!\perp D^c$.

Alternatively, we may directly show the independence between C^c and D^c directly by first calculating $P(C^c \cap D^c)$ as we did in part (d.iii). Then, we compare the result with $P(C^c)P(D^c)$. Both give us $\frac{15}{64}$ and hence $C^c \perp\!\!\!\perp D^c$.

Problem 2. (10 pt) Specify whether each of the following statements is TRUE or FALSE. If it is FALSE, provide your counter-example or explain why it is FALSE.

- (a) For any events A , B , and C , if $A \perp B$ and $B \perp C$, then $A \perp C$.
- (b) If $P(A \cup B) = P(A) + P(B)$, then A and B are disjoint.
- (c) If $A \perp\!\!\!\perp B$, then $P(A) = P(A \cap B^c) + P(A)P(B)$.
- (d) For any events A , B , and C , if $A \perp\!\!\!\perp B$ and $B \perp\!\!\!\perp C$, then $A \perp\!\!\!\perp C$.
- (e) For any events A , B , and C , if $A \perp\!\!\!\perp B$, $B \perp\!\!\!\perp C$, and $A \perp\!\!\!\perp C$, then the events A , B , and C are independent.

Solution:

- (a) **FALSE.** Let $A = C = \{1\}$ and $B = \{2\}$. Then, $A \perp B$ and $B \perp C$. However, A and C are not disjoint.
- (b) **FALSE.** It implies $P(A \cap B) = 0$, which does not mean $A \cap B = \emptyset$. There could be many points in the set $A \cap B$ but all of them has 0 probability.

Remarks:

- As an example, consider $\Omega = \{2, 3\}$ with $P(\{2\}) = 0$ and $P(\{3\}) = 1$. Let $A = \{2, 3\} = \Omega$ and $B = \{2\}$. Then,

$$P(A) = P(A \cup B) = P(\Omega) = 1$$

and

$$P(B) = P(\{2\}) = 0.$$

We then have $P(A \cup B) = P(A) + P(B)$. However, $A \cap B = \{2\} \neq \emptyset$. Hence, A and B are not disjoint.

- You may ask why we need to keep the number 2 in Ω when it does not have any probability. When we consider continuous random variable, we shall see even more counter-intuitive cases where all of the element of Ω has zero probability.
 - It is true that if $A \cap B = \emptyset$, then $P(A \cap B) = 0$ and $P(A \cup B) = P(A) + P(B)$. However, this is not what you are asked to consider in this question.
- (c) **TRUE.** $P(A) = P(A \cap B) + P(A \cap B^c)$. When $A \perp\!\!\!\perp B$, we have $P(A \cap B) = P(A)P(B)$.
- (d) **FALSE.** A counter-example is easily set up by letting $B = \emptyset$ (or $B = \Omega$). In which case, B is automatically independent of any set including A and C . Therefore, the conditions $A \perp\!\!\!\perp B$ and $B \perp\!\!\!\perp C$ are automatically true by this setup for ANY sets A and C . In general, two sets may or may not be independent.

Remark: If you think setting $B = \emptyset$ or $B = \Omega$ seems like cheating, try the following counter-example instead.

Let $\Omega = \{1, 2, 3, 4\}$. Suppose all elements of Ω are equally likely. Set $A = C = \{1, 2\}$ and $B = \{2, 3\}$. Then, $P(A) = P(B) = P(C) = \frac{1}{2}$ and $P(A \cap B) = P[\{2\}] = \frac{1}{4} = P(A)P(B)$. Hence, $A \perp\!\!\!\perp B$. Similarly, because C is the same as A , we also have $C \perp\!\!\!\perp B$. However, $P(A \cap C) = P(A) = 1/2$ is not the same as $P(A)P(C) = P(A)P(A) = 1/4$. Therefore, A and C are not independent.

- (e) FALSE. Pairwise independence is not the same as independence.

Problem 3. (12 pt) Roll a fair six-sided dice five times. Let X_i be the number of dots that show up on the i th roll.

- (a) (4 pt) List all $(X_1, X_2, X_3, X_4, X_5)$ where $X_i \in \{1, 2, 3, 4, 5, 6\}$ such that $X_1 + X_2 + X_3 + X_4 + X_5 = 6$. There should be 5 of these.
- (b) (4 pt) What is the probability that $X_1 + X_2 + X_3 + X_4 + X_5 = 6$?
- (c) (2 pt) What is the probability that $X_1 + X_2 + X_3 + X_4 + X_5 = 10$?
- (d) (2 pt) Given that $X_1 + X_2 + X_3 + X_4 + X_5 = 6$, find the probability that $X_1 = 1$.

Solution:

- (a) Because the minimum value of each X_i is 1, we know that the minimum value for the sum is 5 which occurs when all of the X_i are 1. To make the sum = 6, the only way is to increase the value of one of the X_i from 1 to 2. There are five possible X_i to be increased giving us the list:

$$(2, 1, 1, 1, 1), (1, 2, 1, 1, 1), (1, 1, 2, 1, 1), (1, 1, 1, 2, 1), (1, 1, 1, 1, 2).$$

- (b) For each of the X_i , there are 6 possible values. Because there are 5 of the X_i , the number of possible outcomes is 6^5 . Using the answer from the previous part, the probability that $X_1 + X_2 + X_3 + X_4 + X_5 = 6$ is

$$\frac{5}{6^5} = \frac{5}{7776} = 6.43 \times 10^{-4}.$$

- (c) Let $Y_i = X_i - 1$. Then $Y_i \in \{0, 1, 2, 3, 4, 5\}$. Condition $\sum_{i=1}^5 X_i = 10$ is the same as condition $\sum_{i=1}^5 Y_i = 5$. By the bars and stars argument (or directly apply the formula that we got in class), the number of non-negative integer-valued solutions to $\sum_{i=1}^5 Y_i = 5$ is $\binom{5+5-1}{5-1} = \binom{9}{4} = 126$. Note that these solutions does not directly

require that $Y_i \leq 5$. However, because the sum itself is 5, none of the non-negative integer-valued solution will have any $Y_i > 5$.

The probability that $X_1 + X_2 + X_3 + X_4 + X_5 = 10$ is then given by

$$\boxed{\frac{126}{6^5}} = \frac{7}{432} \approx 0.0162.$$

- (d) Look back at the answer from part (a). There are 4 among the 5 outcomes that $X_1 = 1$. Hence, given that $X_1 + X_2 + X_3 + X_4 + X_5 = 6$, the probability that $X_1 = 1$ is $\boxed{4/5}$.

Problem 4. (6 pt) Suppose that for the Country of Oz, 1 in 1000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 95% of the time. We would like to find the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive.

- (a) (2 pt) What is $P(-|H)$, the conditional probability that a person tests negative given that the person does have the HIV virus?
- (b) (2 pt) Use the law of total probability to find $P(+)$, the probability that a randomly chosen person tests positive. Provide at least 3 significant digits in your answer.
- (c) (2 pt) Use Bayes' formula to find $P(H|+)$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive. Provide at least 3 significant digits in your answer.

Solution:

- (a) Because the test is correct 95% of the time,

$$P(-|H) = P(+|H^c) = \boxed{0.05}.$$

- (b) By the total probability formula,

$$P(+)= P(+|H)P(H) + P(+|H^c)P(H^c) = 0.95 \times 0.001 + 0.05 \times 0.999 \approx \boxed{0.0509}.$$

- (c) Using Bayes' formula, $P(H|+) = \frac{P(+|H)P(H)}{P(+)}$, where the expression for $P(+)$ is given in the previous part. Plugging this expression into the Bayes' formula gives

$$P(H|+) = \frac{0.95 \times 0.001}{0.95 \times 0.001 + 0.05 \times 0.999} \approx \boxed{0.0187}.$$

Thus, even though the test is correct 95% of the time, the probability that a random person who tests positive actually has HIV is less than 2%. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 5. (33 pt) The random variable V has pmf

$$p_V(v) = \begin{cases} \frac{1}{v^2} + c, & v \in \{-2, 2, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

- (a) (5 pt) Find the value of the constant c .
- (b) (2 pt) Find $P[V > 3]$.
- (c) (2 pt) Find $P[V < 3]$.
- (d) (2 pt) Find $P[V^2 > 1]$.
- (e) (3 pt) Sketch $p_V(v)$. Provide as much information on the sketch as you can.
- (f) (3 pt) Sketch $F_V(v)$. Provide as much information on the sketch as you can.
- (g) (4 pt) Let $W = V^2 - V + 1$. Find the pmf of W .
- (h) (3 pt) Find $\mathbb{E}V$
- (i) (3 pt) Find $\mathbb{E}[V^2]$
- (j) (3 pt) Find $\text{Var } V$
- (k) (1 pt) Find σ_V
- (l) (2 pt) Find $\mathbb{E}W$

Solution:

- (a) The pmf must sum to 1. Hence,

$$\frac{1}{(-2)^2} + c + \frac{1}{(2)^2} + c + \frac{1}{(3)^2} + c = 1.$$

The value of c must be

$$c = \frac{1}{3} \left(1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{9} \right) = \boxed{\frac{7}{54}} \approx 0.1296$$

Note that this gives

$$p_V(-2) = p_V(2) = \frac{41}{108} \approx 0.38 \quad \text{and} \quad p_V(3) = \frac{13}{54} \approx 0.241.$$

- (b) $P[V > 3] = \boxed{0}$ because all elements in the support of V are ≤ 3 .
- (c) $P[V < 3] = 1 - p_V(3) = \frac{41}{54} \approx 0.759$.
- (d) $P[V^2 > 1] = \boxed{1}$ because the square of any element in the support of V is > 1 .
- (e)
- (f)
- (g) $W = V^2 - V + 1$. So, when $V = -2, 2, 3$, we have $W = 7, 3, 7$, respectively. Hence, W takes only two values, 7 and 3. the corresponding probabilities are

$$P[W = 7] = p_V(-2) + p_V(3) = \frac{67}{108} \approx 0.62.$$

and

$$P[W = 3] = p_V(2) = \frac{41}{108} \approx 0.38.$$

Hence, the pmf of W is given by

$$p_W(w) = \begin{cases} \frac{41}{108}, & w = 3, \\ \frac{67}{108}, & w = 7, \\ 0, & \text{otherwise.} \end{cases} \approx \begin{cases} 0.38, & w = 3, \\ 0.62, & w = 7, \\ 0, & \text{otherwise.} \end{cases}$$

- (h) $\mathbb{E}V = \frac{13}{18} \approx 0.7222$
- (i) $\mathbb{E}V^2 = \frac{281}{54} \approx 5.2037$
- (j) $\text{Var } V = \mathbb{E}V^2 - (\mathbb{E}V)^2 = \frac{1517}{324} \approx 4.682$.
- (k) $\sigma_V = \sqrt{\text{Var } V} \approx 2.1638$
- (l) $\mathbb{E}W = 5.4815$

Problem 6. (16 pt) The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

	y	2	4	5
x				
1	0.02	0.10	0.08	
3	0.08	0.32	0.40	

- (a) (2 pt) Find the marginal pmf $p_X(x)$.
- (b) (2 pt) Find the marginal pmf $p_Y(y)$.
- (c) (2 pt) Find $\mathbb{E}X$
- (d) (2 pt) Find $P[X = Y]$
- (e) (2 pt) Find $P[XY < 6]$
- (f) (2 pt) Find $\mathbb{E}[(X - 3)(Y - 2)]$
- (g) (2 pt) Find $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)]$
- (h) (2 pt) Are X and Y independent?

Problem 7. (2 pt) A random variables X has support containing only two numbers. Its expected value is $\mathbb{E}X = 5$. Its variance is $\text{Var } X = 3$. Give an example of the pmf of such a random variable.

Solution: We first find $\sigma_X = \sqrt{\text{Var } X} = \sqrt{3}$. Recall that this is the average deviation from the mean. Because X takes only two values, we can make them at exactly $\pm\sqrt{3}$ from the mean; that is

$$x_1 = 5 - \sqrt{3} \quad \text{and} \quad x_2 = 5 + \sqrt{3}.$$

In which case, we automatically have $\mathbb{E}X = 5$ and $\text{Var } X = 3$. Hence, one example of such pmf is

$$p_X(x) = \begin{cases} \frac{1}{2}, & x = 5 \pm \sqrt{3} \\ 0, & \text{otherwise} \end{cases}$$

We can also try to find a general formula for x_1 and x_2 . If we let $p = P[X = x_2]$, then $q = 1 - p = P[X = x_1]$. Given p , the values of x_1 and x_2 must satisfy two conditions: $\mathbb{E}X = m$ and $\text{Var } X = \sigma^2$. (In our case, $m = 5$ and $\sigma^2 = 3$.) From $\mathbb{E}X = m$, we must have

$$x_1q + x_2p = m; \tag{1.1}$$

that is

$$x_1 = \frac{m}{q} - x_2 \frac{p}{q}.$$

From $\text{Var } X = \sigma^2$, we have $\mathbb{E}[X^2] = \text{Var } X + \mathbb{E}X^2 = \sigma^2 + m^2$ and hence we must have

$$x_1^2q + x_2^2p = \sigma^2 + m^2. \tag{1.2}$$

Substituting x_1 from (1.1) into (1.2), we have

$$x_2^2p - 2x_2mp + (pm^2 - q\sigma^2) = 0$$

whose solutions are

$$x_2 = \frac{2mp \pm \sqrt{4m^2p^2 - 4p(pm^2 - q\sigma^2)}}{2p} = \frac{2mp \pm 2\sigma\sqrt{pq}}{2p} = m \pm \sigma\sqrt{\frac{q}{p}}.$$

Using (1.1), we have

$$x_1 = \frac{m}{q} - \left(m \pm \sigma\sqrt{\frac{q}{p}} \right) \frac{p}{q} = m \mp \sigma\sqrt{\frac{p}{q}}.$$

Therefore, for any given p , there are two pmfs:

$$p_X(x) = \begin{cases} 1-p, & x = m - \sigma\sqrt{\frac{p}{1-p}} \\ p, & x = m + \sigma\sqrt{\frac{1-p}{p}} \\ 0, & \text{otherwise,} \end{cases}$$

or

$$p_X(x) = \begin{cases} 1-p, & x = m + \sigma\sqrt{\frac{p}{1-p}} \\ p, & x = m - \sigma\sqrt{\frac{1-p}{p}} \\ 0, & \text{otherwise.} \end{cases}$$

Problem 8. (2 pt) Suppose $X_1 \sim \text{Bernoulli}(1/3)$ and $X_2 \sim \text{Bernoulli}(1/4)$. Assume that $X_1 \perp\!\!\!\perp X_2$.

- (1 pt) Find the joint pmf matrix of the pair (X_1, X_2) .
- (1 pt) Find the pmf of $Y = X_1 + X_2$.

Problem 9. (1 pt) Suppose X and Y are i.i.d. random variables. Suppose $\text{Var } X = 5$ Find $\mathbb{E}[(X - Y)^2]$.

Solution: First, we expand

$$(X - Y)^2 = X^2 - 2XY + Y^2.$$

By linearity of expectation,

$$\mathbb{E}[(X - Y)^2] = \mathbb{E}[X^2] - 2\mathbb{E}[XY] + \mathbb{E}[Y^2].$$

Now,

$$\mathbb{E}[XY] = \sum_{(x,y)} xy p_{X,Y}(x,y) = \sum_x \sum_y xy p_{X,Y}(x,y).$$

By independence between X and Y ,

$$p_{X,Y}(x,y) = p_X(x) p_Y(y)$$

and therefore

$$\mathbb{E}[XY] = \sum_x \sum_y xyp_X(x)p_Y(y) = \sum_x xp_X(x) \sum_y yp_Y(y) = \mathbb{E}X \times \mathbb{E}Y.$$

Moreover, because X and Y are identically distributed, $\mathbb{E}[X^2] = \mathbb{E}[Y^2]$ and $\mathbb{E}X = \mathbb{E}Y$. We can then conclude that

$$\mathbb{E}[(X - Y)^2] = \mathbb{E}[X^2] - 2(\mathbb{E}X)^2 + \mathbb{E}[Y^2] = 2(\mathbb{E}[X^2] - (\mathbb{E}X)^2) = 2 \operatorname{Var} X.$$

In this question, $\operatorname{Var} X = 5$; hence, $\boxed{\mathbb{E}[(X - Y)^2] = 10}$. Remark: Being identically distributed does NOT mean two random variables will always take the same value. For example, toss two coins and let X and Y be the results. Then X and Y are i.i.d. Bernoulli(1/2). This does not mean $X = Y$.